Probit–Based Traffic Assignment: A Comparative Study between Link–Based Simulation Algorithm and Path–Based Assignment and Generalization to Random–Coefficient Approach

Milad Haghani[1], Zahra Shahhoseini[2], Majid Sarvi[3]

Received: 04.05.2014 Accepted: 20.12.2014

Abstract

The problem of path overlapping in network modelling has been one of the main issues to be tackled. Due to its flexible covariance structure, probit model can adequately address the problem. Despite that probit is one of the most appealing choice models, due to the lack of closed form expressions for evaluating choice probabilities; it has not received extensive attention by network modeling researchers. This study is set out to focus on this approach of traffic assignment.

Computational difficulty of application of probit model in the large-scale network equilibrium problem has triggered development of some link-based probit network loading methods which exempt the analyst from generating and maintaining path-flow variables explicitly. The bias of these heuristic link-based methods has not been studied so far. This contribution primarily focuses on investigation of such potential bias in link-based probit assignment methods. In this research, this bias for a certain simulated link-based method is empirically considered and investigated through comparison with path-based probit equilibrium solution.

Capable of representing utility correlation and heteroscedasticity, probit model has always been one of the most theoretically attractive models for representing route choice behavior. However, this soundness of theory could further be enhanced through combining the ideas of probit and random-coefficient modeling which enables the analyst to capture random taste heterogeneity over travelers as well.

Keywords: Probit Model, multivariate normal distribution, Monte Carlo Simulation, random-coefficient choice models, link-based and path-based traffic assignment,

Corresponding author E-mail: milad.haghani@monash.edu

1- Ph.D. Candidate, Department of Civil Engineering, Institute of Transport Studies, Monash University, Melbourne, Australia

2- Ph.D. Candidate, Department of Civil Engineering, Institute of Transport Studies, Monash University, Melbourne, Australia

3- Associate Professor, Department of Civil Engineering, Institute of Transport Studies, Monash University, Melbourne, Australia

1. Introduction

As the final stage of traditional four-step transportation planning models, traffic assignment (TA) plays a crucial role in prediction of flow as well as travelers' response to any changes in the system and thereby making optimal efficient decisions by analysts. In this sense, there has been a considerable effort to enhance precision of route choice modeling. Probabilistic approach of network modeling was primarily developed to address and represent randomness in travelers' behavior and to provide a more realistic and flexible basis for modeling transportation networks.

Since the development of the notion of stochastic user equilibrium (SUE) concept by Daganzo and Sheffi [1977] different types of choice models were employed in this context. Although there has been limited studies in probabilistic network loading procedures prior to the introduction of SUE, their new formulation as a generalization to traditional user equilibrium (UE) criteria as well as their comprehensive discussion on enumerating the deficiencies of multinomial logit (MNL) model [Dial, 1971] for route choice modeling purposes triggered considerable further studies in this area.

Theoretically, multinomial probit (MNP) model is one of the most appealing choice models enjoying sound underlying assumptions which is able to not only capture correlation among random utilities but also is capable of representing heteroscedasticity property. In other words, probit model is potentially able to cover the main drawbacks of MNL model in TA context. Due to the lack of analytical closed-form expressions for calculating choice probabilities, however, this approach of TA has not received sufficient attention by comparison with logit-family models. Nevertheless, to alleviate computational expenditure of this model, some studies have been dedicated to development of heuristic link-based algorithms for probit network loading [Sheffi and Powell, 1981; Maher 1992]. This approach was indeed primarily aimed to exempt the analyst from producing and maintaining path-flow variables in solving network equilibrium problem. But, along with the advances in computing technology as well as development of efficient path generation algorithms, there has been a growing recognition of path-based approach [for example, see Bekhor et al. 2008; Prato and Bekhor, 2006; Prato and Bekhor, 2007; Bekhor et al. 2006]. Furthermore, there is now a general consensus that path-based approach of TA allows application of more advanced choice models.

Due to the presence of the aforementioned link-based algorithms, there has not been sufficient impetus for investigating path-based probit TA. In particular, to the best of our knowledge, the bias problem for these link-based methods has not been studied formerly. This study intends to investigate this problem for the Monte Carlo simulation algorithm proposed by Sheffi and Powell [1981] through numerical comparison with the path-based probit SUE solution.

Potential capacity of random-utility theory also allows further enhancement of theoretical attractiveness of probit model through relaxation of the remaining restrictive assumptions, in particular the "fixed-coefficient" assumption. Although there has been considerable studies in mixed MNL conducted in different areas of choice modeling (including route choice but in a very limited manner [Bekhor et al. 2002]), to the best of our knowledge, this approach has not yet been generalized to mixed probit in the literature. In addition to the previously-mentioned theoretical advantages of ordinary probit model, a mixed (random-coefficient) probit can also take into account the fact that different travelers have different sensitivities with respect to the attributes of different paths, and in this sense there can be a random taste variation over the population of decision makers. This representation of random taste heterogeneity for mixed probit can potentially improve the precision of TA in describing behavior inasmuch as it enjoys a more generalized and relaxed underlying assumption than the traditional probit. This approach is formulated and applied to an illustrative network in this study.

The remaining parts of the paper are organized as follows. Section 2 has been dedicated to a literature review. In section 3, theoretical backgrounds for probit and mixed probit models are provided. Section 4 defines the methodology applied to conduct the comparative part of the study. In section 5, the bias of the link-based probit as well as application of the proposed mixed probit model will be investigated on an illustrative network together with some sensitivity analyses with respect to contributing parameters of the problem. Section 6 summarizes the paper and findings and proposes future research directions for further studies.

2. Literature Review

Preliminary studies in probabilistic approach of transportation network loading can be attributed to Burrell [1968] and Von Falkenhausen [1966] both constructed upon the notion of Monte Carlo simulation. Dial [1971] also proposed an efficient link-based network loading algorithm, supported by formal mathematical proof, which without requiring explicit path generation, assigns trip demands to the subsets of the universal paths in accordance with MNL formula. The algorithm, also known as STOCH in the literature, provoked many subsequent studies investigating shortcomings of the STOCH algorithm or making attempts to propose improvements or generalizations [Daganzo and Sheffi, 1977; Florian 1974; Florian and Fox, 1976; Tobin, 1977; Fisk, 1980]. Some other researches also studied the theoretical shortcomings of STOCH algorithm criteria for path filtration and the concept of "efficient paths" in this algorithm [Bell, 1995; Akamatsu, 1996].

The major deficiency of the MNL model in network modeling purposes arises from the fact that this model does not allow the topology of the network to be reflected in evaluation of choice probabilities. In other words, given the values of representative utilities, MNL choice probabilities are definite no matter what the geometrical structure of network is. This problem gives rise to unrealistic overestimation of the flow on highly overlapped paths and underestimation of the flow on more independent paths. In addition, due to the simple diagonal homoscedastic structure of MNL covariance matrix, absolute values of utilities, similar to correlations, have no reflection to calculation of choice probabilities. In other words, when MNL is applied as the route choice model, a (for example) 5-minutes difference of travel time in a binary route choice situation, has the same effect in a short trip as it has in a long journey. Still, most of the researcher's attention in this context has been devoted to proposing solutions for the former shortcoming, that is to say, the problem of path overlapping. Miscellaneous models has been proposed from simple modifications of MNL model [Cascetta et al. 1996; Ben-Akiva and Bierlaire, 1999; Bovy et al. 2008] to generalized extreme value-type models [Prashker and Bekhor, 1998]. Further details of the different types of logit-based STA models can also be found in the study conducted by [Haghani et al. 2013].

Probit model, however, is theoretically capable enough to address both correlation and heteroscedasticity problems. Application of MNP in TA was first formulated by Daganzo and Sheffi [1977] and further developed by Sheffi and Powell [1981] where they proposed a relevant covariance structure as well as a heuristic Monte Carlo link-based method for probit assignment. They formulated the MNP covariance structure by pertinently relating the elements of the matrix to network topology. Nonetheless, Due to their proposed link-based approach, they did not need to explicitly evaluate and maintain covariance matrices. Structuring the covariance matrix for probit-based TA is crucial from modeling purposes since due to the very large number of alternatives in route choice problem it would not be neither practical nor possible to estimate all elements of free (unrestricted) covariance matrices from a choice data set. This way, the number of covariance-related estimation parameters is substantially reduced.

Maher [1992] and Maher and Hughes [1997] proposed another link-based method for probit-based network loading as well as a convex combinations algorithm based upon it for solving stochastic user equilibrium (SUE) problem. The steps of the algorithm and its node-by-node progression through network are highly analogous to the STOCH algorithm. Similar to the STOCH algorithm, link probabilities are evaluated through "forward-step" progression phase and network is loaded in "backward-step" phase. The major difference is that normal distribution is assumed for "link utilities", and rather than having closed form expressions for evaluating "link probabilities", Clark method [Clark, 1961] is applied for approximation of these probabilities. In this sense, the algorithm can be regarded as a modified version of the STOCH algorithm for probit model. The main advantage of this method, in addition to the efficiency as a result of its link-based structure, is that during the progression of the algorithm through the network, some information is gathered which can be utilized for evaluation of the general SUE objective function formulated by Sheffi and Powell [1982] and its derivatives to approximate the optimal step sizes in the convex combinations procedure. This property indeed adds further efficiency to the algorithm. Despite these advantages, however, the adequacy of precision of Clark approximation for this application needs to be scrutinized as Horowitz et al. [1982] discussed that the precision of this approximation method significantly varies from situation to situation. But more importantly, whether the algorithm actually produces probit flows is arguable. Contrary to the STOCH algorithm, this algorithm has not been accompanied by any formal analytical proof. Further, as stated by Akamatsu [1996], MNL assignment contains an implicit assumption called the "Markovian property" stating that "the pattern of route choice from an intermediate node to the destination is independent of the pattern of route choice from the origin to that node". This is exactly the characteristic that allowed development of the STOCH algorithm for MNL network loading. But, the presence of this property for MNP assignment has neither been proved nor investigated, and it seems to be hardly likely for this property to be valid for MNP as well. In other words, it seems that the intrinsic Markovian property of the STOCH algorithm has exogenously been imposed to MNP assignment in the method proposed by Maher [1992].

Yai and co-authors [1997] estimated a structured-covariance MNP model for train route choice based on experimental data. Clark and

Watling [2000] conducted a sensitivity analysis of the MNP-SUE flow with respect to factors such as demand and utility coefficients. Zhang et al. [2008] utilized MNP for dynamical TA.

3. Theoretical Background

This section aims to provide a quick review on the theoretical concept of MNP and mixed MNP choice modeling. Simulation methods for evaluation of probabilities are also discussed and the specification of the model for route choice is presented.

3.1 Multinomial Probit

Let us consider a typical trip maker n who faces a set including K paths for traveling between the origin and destination of his journey. Let us also represent the utility perceived by the person from a typical path k by U_{nk}. This utility consists of a systematic part, V_{nk}, as well as a random disturbance term, ε_{nk} (Eq. 1), which the latter contains all the factors contributing to and affecting the decisions but not considered by the analyst in the systematic part. The probability distribution assumed for the random term plays an important role from modeling perspective as different probability distributions would lead to different choice models. For MNP model, error terms are assumed as a vector of random variables distributed multivariate (or joint) normal (MVN) with mean vector of 0 and an arbitrary covariance matrix Σ_n (Eqs. 2 and 3).

$$U_{nk} = V_{nk} + \varepsilon_{nk} \quad \forall\, n, \forall k \tag{1}$$

$$\boldsymbol{\varepsilon}_n \sim MVN(\mathbf{0}, \boldsymbol{\Sigma}_n) \quad \forall n \tag{2}$$

$$\boldsymbol{\Sigma}_n = \left[Cov(\varepsilon_{kn}, \varepsilon_{jn})\right];\ k = 1, \dots, \mathcal{K}\ , j = 1, \dots, \mathcal{K} \tag{3}$$

Theoretically speaking, the assumption of joint normal distribution is a sound fundamental assumption, notably in comparison with the assumption of identical and independent Gumbel distributions for MNL. This assumption is also supported by multivariate central limit theorem [Rice, 1995]. This flexible assumption also allows representation of substitution patterns of any type [Train, 2009]. In other words, by specification of the covariance matrix, the analyst is able to take into account heteroscedasticity as well as any type of correlation structure between utilities. In general, each covariance matrix is always symmetric and non-negative definite (or equivalently, positive semi-definite) [Eaton, 1983]. A MVN distribution is called non-degenerate whenever its covariance matrix is (strictly) positive definite. For a non-degenerate MVN distribution, there is a closed-form probability distribution function. This closed form for distribution of error disturbances specified in Eq. 2 is as Eq. 4 where $|\Sigma_n|$ signifies determinant of Σ_n. The probability for alternative k to provide the most utility and hence to be chosen by person n is also given by the multivariate integral in Eq. 5. The integral generally cannot be solved analytically, and accordingly, simulation or approximation methods are required for evaluation of MNP probabilities. Different approximate methods

$$f_{\boldsymbol{\varepsilon}_n}(\varepsilon_{1n}, \varepsilon_{2n}, \dots \varepsilon_{\mathcal{K}n}) = \left((2\pi)^{\mathcal{K}} |\boldsymbol{\Sigma}_n|\right)^{-1/2} \exp\left(-\frac{1}{2}\boldsymbol{\varepsilon}_n^T . \boldsymbol{\Sigma}_n^{-1} . \boldsymbol{\varepsilon}_n\right) \tag{4}$$

$$P_{kn} = \int_{U_1=-\infty}^{U_k} \dots \int_{U_k=-\infty}^{+\infty} \dots \int_{U_{\mathcal{K}}=-\infty}^{U_k} \left((2\pi)^{\mathcal{K}} |\boldsymbol{\Sigma}_n|\right)^{-1/2} \exp\left(-\frac{1}{2}(\boldsymbol{U}_n - \boldsymbol{V}_n)^T . \boldsymbol{\Sigma}_n^{-1} . (\boldsymbol{U}_n - \boldsymbol{V}_n)\right) dU_1 \dots dU_{\mathcal{K}} \tag{5}$$

have been proposed for this paper in the literature upon which Rosa [2003] has made a comprehensive review. Different simulation methods are also available [Train, 2009] among which we have implemented the accept-reject method because of its simplicity.

According to the accept-reject method, the major problem for simulation of MNP probabilities is to generate random draws from MVN distributions. In general, to generate a K-dimensional vector ν from a MVN distribution with mean μ and covariance Σ, the following steps should be taken [Eaton, 1983]:

1- Produce a vector of K standard normal random draws: $z^T=z_1,\ldots,z_K$.

2- Find real matrix A in such a way that: $AA^T=\Sigma$.

3- Set $\nu=\mu+Az$.

Therefore, the main problem is to find a matrix A which when multiplied by its transpose, produces the covariance matrix. In the case that the covariance is positive definite, the matrix A can easily be obtained by Cholesky factorization method [Horn, 1985]. In the general case of positive semi-definiteness, however, this can be done by the method called spectral decomposition (or eigendecomposition) [Horn, 1985]. Simply, in this method, A can be calculated by Eq. 6 in which M is an orthogonal matrix for which the jth column is the jth eigenvector of covariance matrix Σ and Λ is a diagonal matrix containing the corresponding eigenvalues.

$$A=M\Lambda^{(1/2)} \tag{6}$$

Our specification of covariance matrix for path utilities corresponding to a particular origin-destination (O-D) pair is compatible with the implicit specification of Sheffi and Powell [1981]. The structured covariance matrix is given in Eq. 7 in which l_k represents the length of path k and l_kj denotes the common length of paths k and j. According to the formulation of Sheffi and Powell [1981], the path length can be substituted by free-flow path travel time as well. The parameter β is to be calibrated and can be interpreted as the variance of utility perceived from a segment of path with one unit length. So, this specification assumes that the variance of error (or utility) for each path is proportionate with the length of that path and the covariance between random utilities of each pair of paths is proportionate with their shared segment length. The less the value of β, the more similar is the flow pattern to the traditional user equilibrium (UE) solution.

In univariate case, the systematic part of utility can also be specified as Eq. 8 in which t_k denotes path travel time and α is its utility coefficient. In accordance with the study of Sheffi and Powell [1981] we have set α =1. Additionally, to compare the link-based and path-based MNP assignments, we have examined different values of β= 1, 2, 3, 4 in numerical experiments.

$$\boldsymbol{\Sigma} = \beta \begin{pmatrix} l_1 & l_{12} & \cdots & l_{1\mathcal{K}} \\ l_{21} & l_2 & \cdots & l_{2\mathcal{K}} \\ \vdots & \vdots & \ddots & \vdots \\ l_{\mathcal{K}1} & l_{\mathcal{K}2} & \cdots & l_{\mathcal{K}} \end{pmatrix} \tag{7}$$

$$V_k = -\alpha t_k \quad \forall k \tag{8}$$

3.2 Mixed Multinomial Probit

In classical MNP, the coefficient α is assumed to be a fixed value for the whole population of decision makers indicating that all travelers have a similar degree of sensitivity to travel time in their route choice. The more generalized random-coefficient (or mixed) specification allows the modeler to assume a probability distribution for the coefficients. This way, he/she can take into consideration the fact that different decision makers have different tastes and as a result, there can be a distribution of tastes over the population of travelers. Therefore, the mixed MNP modeling approach addresses random taste variation as well as the aforementioned theoretical benefits of the classical MNP model.

To generalize our route choice model specified in Eqs. 7 and 8 to random-coefficient model, a probability distribution should be attributed to α. Theoretically, different distributions are applicable. Two common assumptions, however, are normal and log-normal distributions [Train, 2009] at least for mixed MNL modeling which is the only available published random coefficient modeling approach in the choice modeling literature. Both the two normal and log-normal distributions serve our modeling purpose. But, in cases such as route choice where the modeler has a prior assumption about the sign of a certain parameter, log-normal distribution provides a theoretical benefit since it takes only positive values. While normal distribution allows both negative and positive values, for variables such as travel time which the analyst believes that all decision makers are trying to avoid, log-normal distribution provides more theoretical soundness. In this study and in implementation of mixed MNP model, we assumed α to be distributed log-normally as Eq. 9, where m and s are parameters of the distribution, that is, mean and standard deviation respectively. For numerical implementations in the following section, different distributions will be examined setting m=1 and varying the value of the standard deviation. We have also set β=1 for mixed model.

$$\ln\alpha \sim N(m,s^2) \tag{9}$$

In order to calculate mixed MNP probabilities, a further simulation procedure is required. Theoretically, mixed MNP probabilities, P □_ kn, are the solution of the integral given in Eq. 10 which is simply the integral of the classical MNP probabilities over the density probability function of α. As this integral cannot be analytically solved as well, it has to be simulated. Simulation of the integral, however, is straightforward. One should draw random samples from the density function of α, then simulate MNP probability for each random draw (as explained previously), repeat this procedure for many times and finally average the results over the simulation iterations.

$$\hat{P}_{kn} = \int_{\alpha} P_{kn}\, f(\alpha) d\alpha \tag{10}$$

4. Methodology Definition

In order to conduct the comparative study introduced in the previous sections between the Monte-Carlo simulation link-based MNP traffic assignment and its path-based counterpart approach, we will apply both methods on a small illustrative network which allows us first, to conduct the path-based assignment on the full sets of paths (exhaustive path enumeration), and second, allows us to represent the detailed values of link (or path) flows in manageable-sized Tables. Then we will provide some statistical comparisons to further clarify the potential bias that application of the link-based simulated method might engender. In other words, we are regarding the values of the path-based probit flows as the "true" probit values (as they have resulted from application of MNP in its standard fashion) and then we will investigate the extent to which the heuristic link-based method is capable of reproducing those flows, and through which we will answer the question that how accurate the link-based algorithm is.

5. Numerical Examples

Numerical investigations and comparisons have been conducted on the illustrative network proposed by Nguyen and Dupuis [1984] (Figure 1). The network was chosen as the basis of comparison since its manageable size facilitates representation of link and path flows as well as sensitivity analyses. The proposed logit route choice models are applied in the pedagogical network of Nguyen and Dupuis (1984) with 13 nodes, 19 links, and 4 O-D pairs (Figure 1). O-D pairs of (1,2), (1,3), (4,2) and (4,3) have, respectively, travel demand of 400,800,600 and 200 units. The volume-delay function of $T_a(x_a)=\alpha_a+\beta_a x_a$, in which $(T)_a$ is travel time, and x_a is flow on link a, is utilized. The values of parameters α_a and β_a are presented in Table 1.

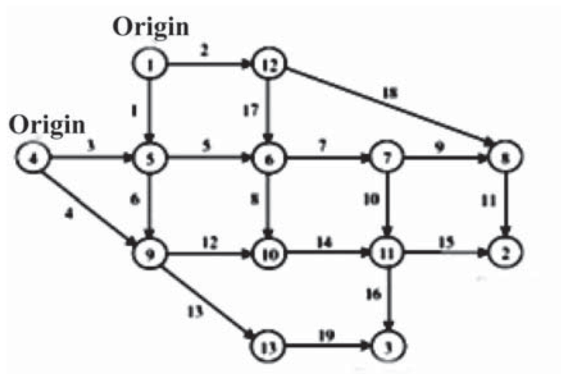


Figure 1. Illustrative network

Table 1. Parameters of link volume-delay functions for illustrative network

Link	Start Node	End Node	α_a	β_a
1	1	5	7	0.0125
2	1	12	9	0.01
3	4	5	9	0.01
4	4	9	12	0.005
5	5	6	3	0.0075
6	5	9	9	0.0075
7	6	7	5	0.0125
8	6	10	13	0.005
10	7	11	9	0.0125
11	8	2	9	0.0125
12	9	10	10	0.005
13	9	13	9	0.005
14	10	11	6	0.0025
15	11	2	9	0.005
16	11	3	8	0.01
17	12	6	7	0.0125
18	12	8	14	0.001
19	13	3	11	0.001

5.1 Comparison of Link-based and Path-Based Probit Assignment

Table 2 illustrates the result of TA for path-based probit as well as the link-based Monte Carlo method proposed by Sheffi and Powell. Equilibrium link flows were obtained by path-based and link-based versions of the method of successive averages (MSA) [Sheffi and Powell, 1981] respectively and by 1000 iterations of the algorithm. To mitigate the stochastic effects of simulation to the least possible extent, choice probabilities were obtained by 10,000 number of simulation iteration which can be regarded as a fairly large number for this implementation. Also, as already mentioned, in order to eliminate the effect of the generated choice set size on the result of comparison between path-based and link-based assignment, an exhaustive path enumeration approach has been applied, that is to say, all possible paths between each pair of O-D have been produced and used in the path-based TA procedure. For larger-scale networks, however, utilizing a method for generating subsets of the universal sets of paths will be inevitable, the discussion of which is outside the scope of this contribution. [Prato and Bekhor, 2006; Prato and Bekhor, 2007; Haghani et al. 2014]. As Table 2 shows, for each particular value of β, the SUE flows obtained by the heuristic link-based method proves to be considerably different from the MNP-SUE flows. In order to facilitate the comparison, we also have calculated the absolute value of the percentage of relative difference between the corresponding flows. Results demonstrate that the average of this difference for β=1,2,3,4 is roughly 20%, 26%, 29% and 31% respectively, and the maximum would amount to 61%, 67%, 69%, and 78%. This result has also been illustrated in Figure 2. Table 3 also represents the MNP-SUE path flows along with equilibrium path times which are obviously associated with the path-based algorithm.

Table 2. Comparison of path-based and link-based probit link flows

Link Number	β=1			β=2			β=3			β=4		
	Path-Based	Link-Based	RD (%)	Path-Based	Link-Based	RD (%)	Path-Based	Link-Based	RD (%)	Path-Based	Link-Based	RD (%)
1	681	704	3.4	691	703	1.7	696	702	0.9	700	702	0.3
2	519	496	4.4	509	497	2.4	504	498	1.2	500	498	0.4
3	262	102	61.1	309	103	66.7	338	104	69.2	358	105	70.7
4	538	698	29.7	491	697	42.0	462	696	50.6	442	695	57.2
5	527	440	16.5	556	441	20.7	574	441	23.2	587	441	24.9
6	416	366	12.0	444	365	17.8	460	365	20.7	472	365	22.7
7	430	354	17.7	459	356	22.4	478	357	25.3	492	357	27.4
8	309	182	41.1	340	183	46.2	358	183	48.9	371	184	50.4
9	163	102	37.4	179	103	42.5	188	104	44.7	193	105	45.6
10	266	252	5.3	280	253	9.6	291	253	13.1	299	252	15.7
11	470	502	6.8	444	503	13.3	428	503	17.5	417	504	20.9
12	453	498	9.9	462	497	7.6	468	496	6.0	473	495	4.7
13	501	566	13.0	473	565	19.5	455	565	24.2	441	565	28.1
14	762	680	10.8	803	679	15.4	827	679	17.9	843	679	19.5
15	530	498	6.0	556	497	10.6	572	497	13.1	583	496	14.9
16	499	434	13.0	527	435	17.5	545	435	20.2	559	435	22.2
17	212	97	54.2	244	98	59.8	263	99	62.4	276	99	64.1
18	307	400	30.3	265	399	50.6	241	399	65.6	224	399	78.1
19	501	566	13.0	473	565	19.5	455	565	24.2	441	565	28.1

Note: RD (%) indicates the (absolute) percentage of the relative difference

Table 3. Equilibrium path flows and path times for MNP assignment

O-D	Path Number	Link Sequence				β=1		β=2		β=3		β=4	
						Path Flow	Path Time	Path Flow	Path Time	Path Flow	Path Time	Path Flow	Path Time
1-2	1	2-18-11				307	46.1	265	45.3	241	44.8	224	44.4
	2	1-5-7-9-11				21	54.8	27	55.3	30	55.7	32	55.9
	3	1-5-7-10-15				17	56.8	20	57.8	22	58.5	24	59.0
	4	1-5-8-14-15				23	56.6	29	57.3	32	57.7	34	58.0
	5	1-6-12-14-15				10	59.5	20	60.1	25	60.4	30	60.7
	6	2-17-7-9-11				6	56.1	11	56.7	14	57.0	17	57.2
	7	2-17-7-10-15				7	58.2	11	59.2	14	59.8	16	60.2
	8	2-17-8-14-15				10	57.9	17	58.6	21	59.0	24	59.3
1-3	9	1-6-13-19				350	55.2	334	55.1	322	55.0	314	54.9
	10	1-5-7-10-16				104	58.1	93	59.3	90	60.1	88	60.6
	11	1-5-8-14-16				115	57.9	110	58.8	107	59.3	106	59.7
	12	1-6-12-14-16				41	60.8	58	61.6	67	62.0	73	62.4
	13	2-17-7-10-16				88	59.5	95	60.7	98	61.4	100	61.9
	14	2-17-8-14-16				102	59.3	111	60.1	116	60.6	119	61.0
4-2	15	4-12-14-15				360	46.5	327	46.6	307	46.6	293	46.6
	16	3-5-7-9-11				136	50.9	141	51.8	143	52.4	144	52.8
	17	3-5-7-10-15				44	52.9	51	54.3	55	55.1	58	55.8
	18	3-5-8-14-15				51	52.7	63	53.7	69	54.4	73	54.9
	19	3-6-12-14-15				8	55.6	18	56.5	25	57.1	31	57.5
4-3	20	4-13-19				145	42.2	127	41.5	117	41.1	109	40.8
	21	4-12-14-16				32	47.8	37	48.0	39	48.2	39	48.3
	22	3-6-13-19				6	51.3	12	51.5	16	51.6	18	51.7
	23	3-5-7-10-16				7	54.2	10	55.8	11	56.7	13	57.4
	24	3-5-8-14-16				8	54.0	11	55.2	14	56.0	15	56.5
	25	3-6-12-14-16				1	56.9	3	58.0	4	58.7	6	59.2

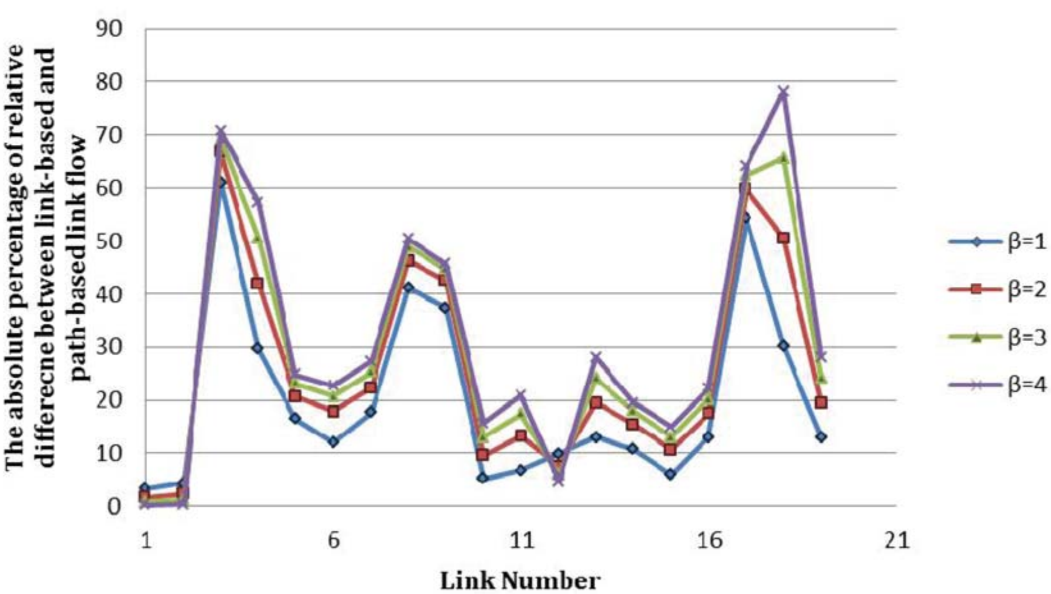


Figure 2. Comparing the link-based and path-based MNP link flows in terms of the absolute value of percentage of relative difference.

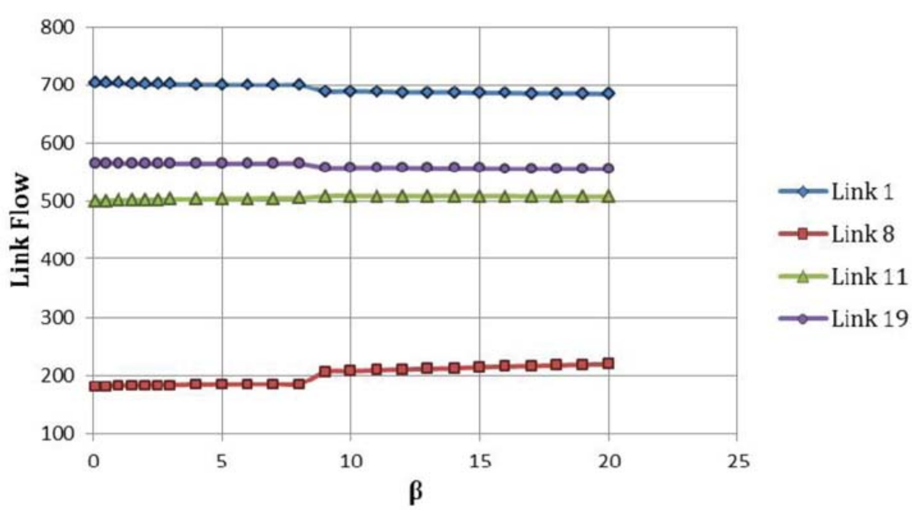
5.2 Sensitivity Analysis

Sensitivity analyses of equilibrium flows will be made in this section with respect to different parameters of the problem including the value of the parameter β, number of simulation iterations and level of congestion.

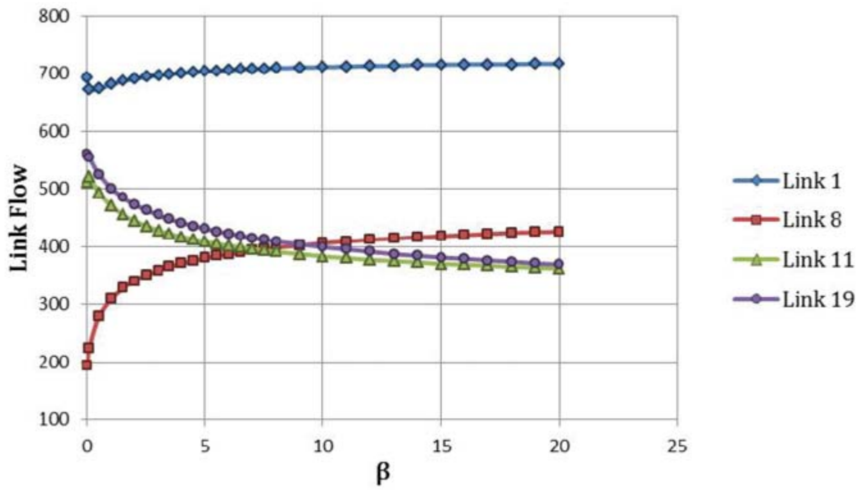
5.2.1 Sensitivity to β

In order to investigate the effect of β on prediction of MNP-SUE flow, a sensitivity analysis was conducted. Figure 3 illustrates variation of equilibrium link flows for some certain links of our example network when β is varied from 0 to 20 for both path-based and link-based approaches. As can be seen, while the path-based equilibrium solution is significantly dependent on the value of β, particularly on an interval of β=0 to β=5, the result of the link-based algorithm is only slightly sensitive to this parameter.

Figure 4 depicts the sensitivity in terms of path flows connecting a particular O-D pair (O-D pair 4-2), obviously for the path-based method. This additional figure aims to graphically demonstrate the fact that, contrary to what has been mentioned in some previous studies [for example, Prashker and Bekhor, 2000], as the variance of errors approaches to infinity, SUE path flows corresponded with each O-D pair do not tend to equal values. The proposition has mistakenly been stated in some published studies that when the variance of error terms tends to infinity, based on all stochastic models forecast, travelers become indifferent to various paths and path utilities are perceived equally. But, as the Figure 5 shows, this statement is not true for MNP model. In general, the proposition is only valid for the simple MNL model due to its diagonal covariance structures. From the perspective of other models which are able to represent correlation, path probabilities and as a result, path flows are determined by the covariance structure and necessarily do not lead to equal values when variance approaches to infinity.



(a) Link-Based Probit.



(b) Path-Based Probit.

Figure 3. Sensitivity of probit link flows to the value of β.

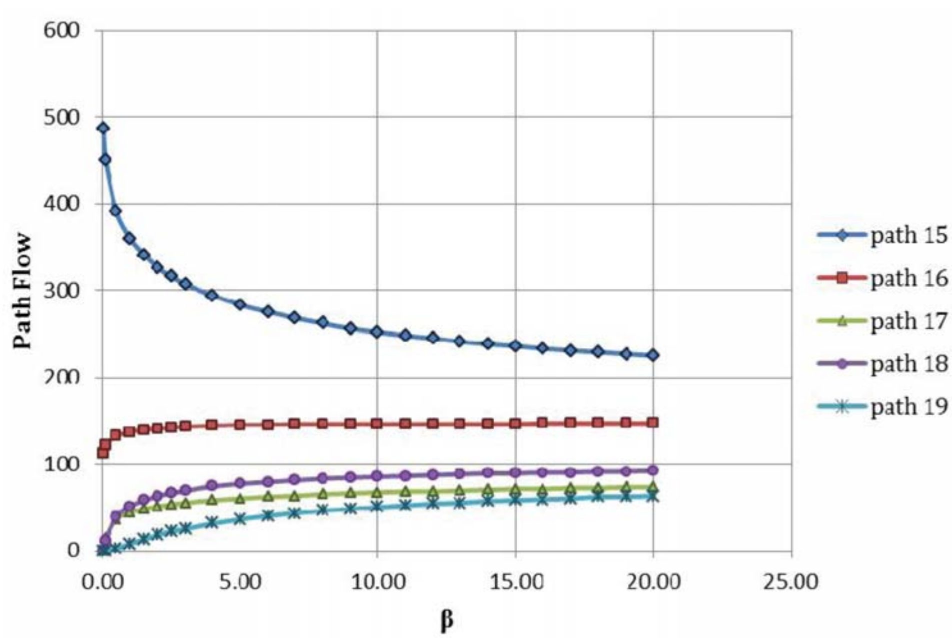


Figure 4. Sensitivity of probit path flows to the value of β.

5.2.2 Sensitivity to the Number of Simulation Iterations

One important factor that significantly affects computational expenditure of probit-based TA is the number of simulation iterations which the analyst has to set to obtain probit probabilities. Table 3 represents the sensitivity of the flow to the number of simulation iterations for both path-based and link-based methods. The results are correspondent with β=1. As can be observed, while path-based probit simulation approaches to its long-run solution quite rapidly, the result of the link-based method is considerably dependent on the number of simulations. Therefore, as a computational benefit, the path-based method does not basically require very large number of iterations in Monte Carlo procedures while the link-based algorithm does. This property can indeed be considered as a compensation for the lower degree of efficiency for path-based approach in comparison with the link-based algorithm.

5.2.3 Sensitivity to the Level of Congestion

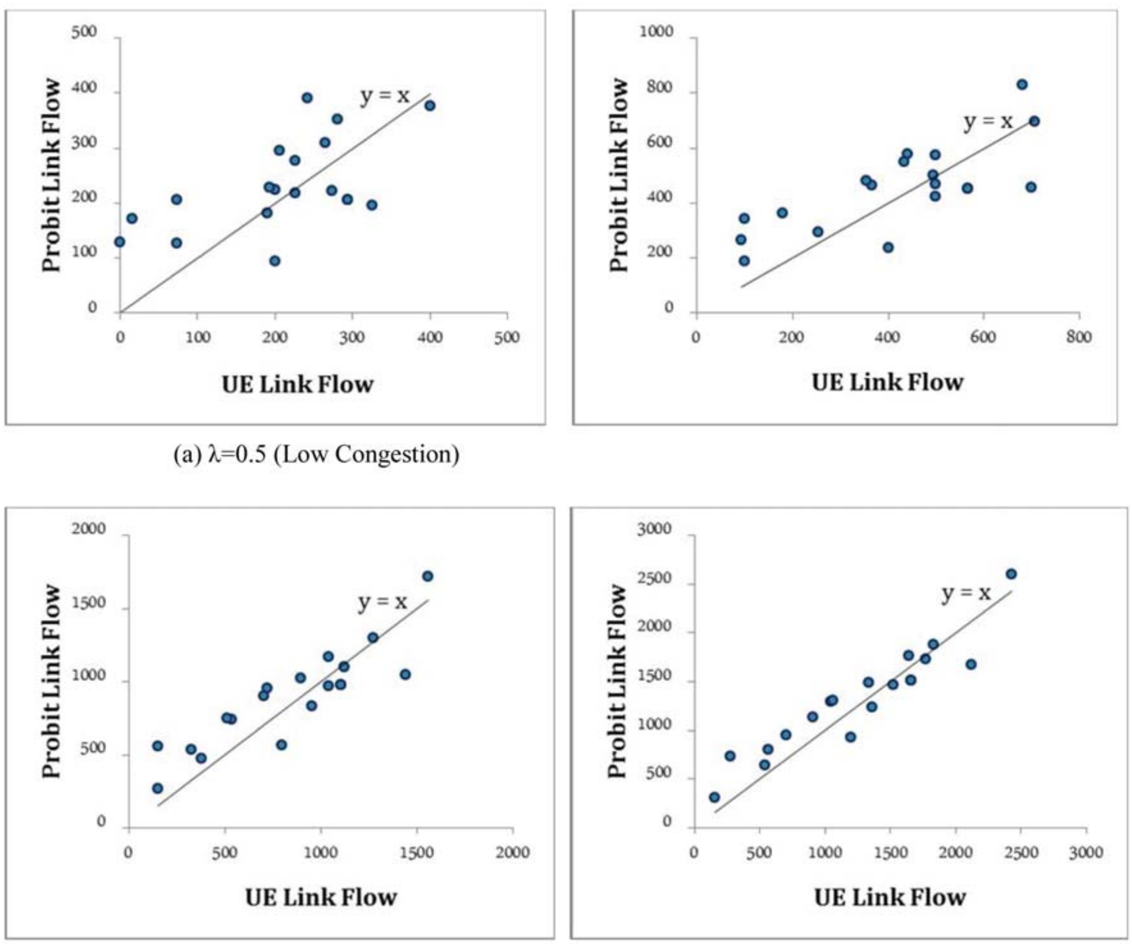
The level of congestion in the network is another influential factor in prediction of STA models including MNP model. As discussed in the literature, when the network becomes more congested, SUE flow becomes more and more similar to UE. In other words, for highly congested networks, prediction of stochastic models is more similar to UE flow pattern

Table 4. Sensitivity of the MNP link flows to the number of simulation iterations

Link Number	Path-Based					Link-Based				
	10 Iterations	50 Iterations	100 Iterations	1000 Iterations	10000 Iterations	10 Iterations	50 Iterations	100 Iterations	1000 Iterations	10000 Iterations
1	680	698	697	698	698	672	684	688	699	704
2	520	502	503	502	502	528	516	512	501	496
3	262	344	344	344	344	187	140	129	106	102
4	538	456	456	456	456	613	660	671	694	698
5	527	578	577	577	577	481	457	451	442	440
6	415	464	464	464	464	378	367	366	363	366
7	429	483	482	482	482	393	367	365	355	354
8	311	362	362	362	362	248	214	203	190	182
9	163	189	189	189	189	140	122	117	105	102
10	266	294	293	293	293	253	246	247	249	252
11	471	425	425	425	425	508	513	513	504	502
12	453	468	469	469	469	450	469	476	494	498
13	500	451	451	451	451	541	558	561	563	566
14	764	830	831	831	831	697	683	679	684	680
15	529	575	575	575	575	492	487	487	496	498
16	500	549	549	549	549	459	442	439	437	434
17	212	266	267	267	267	159	124	117	102	97
18	307	236	236	236	236	368	392	395	399	400
19	500	451	451	451	451	541	558	561	563	566

than for lightly congested networks [Daganzo and Sheffi, 1977; Sheffi and Powell, 1981]. The reason is clear. When the level of travel demand and as a result, link travel times increase, the systematic part of utilities plays a more important role in determination of probabilities and in asymptotic case, the influence of error disturbances pales into insignificance and predicted flow pattern tends to UE.

Although analytically understandable, the aforementioned problem is important and needs to be numerically investigated in the sense that it can in fact question the logical necessity for application of STA models for highly congested networks. Numerical examinations seems to be required to uncover that whether for highly congested networks SUE models are indeed unnecessary. To quantitate the problem, we considered different degrees of congestion level by multiplying the base demand vector of our example network by factors λ=0.5, 1, 2 and 3. As the base demand of the network can be considered moderate, λ=3 can be regarded as a very high level of congestion. As Figure 5 shows, results confirm the prior assumption as to the increase of similarity between UE and MNP-SUE flows. Still, they also show that even for very large levels of congestion, there are significant and meaningful differences between the two approaches.



(a) λ=0.5 (Low Congestion)

Figure 5. Sensitivity of probit flow to the level of congestion

5.3 Mixed Probit SUE

Mixed MNP-SUE flows were obtained for different probability distributions assumed for the coefficient of travel time. Probit probabilities as well as mixed probabilities each were calculated by 1000 times of random draws. Number of MSA replications was also set to 1000. Equilibrium link and path flows have been illustrated in Tables 5 and 6, respectively. As the results show, regardless of the value of the standard deviation, transition from classical to random-coefficient MNP has made a significant contribution to prediction of flow. Although there are differences between the results of applying mixed MNP assignment with different values of standard deviation, one can say that different values of standard deviation has led to by-and-large similar flow patterns. By comparison, however, there is a remarkable difference between classical MNP-SUE and mixed MNP-SUE predictions. Comparing the figures presented in Tables 3 and 6 one can conclude that the mixed approach has produced a more disperse flow pattern with higher degrees of stochasticity in route finding than non-mixed MNP. In other words, non-shortest paths, are receiving bigger shares of the total demand based upon mixed-MNP prediction than the classical version. This more intensity of stochasticity was anticipaTable inasmuch as the mixed model captures an additional source of randomness in route choice behavior.

Table 5. Mixed MNP link flows

Link Number	m=1.0 s=0.2	m=1.0 s=0.4	m=1.0 s=0.6
1	708	707	706
2	492	493	494
3	399	393	386
4	401	407	414
5	613	606	597
6	494	494	495
7	520	515	508
8	394	388	381
9	203	201	197
10	317	314	311
11	394	397	398
12	481	481	482
13	414	420	427
14	875	869	863
15	606	603	602
16	586	580	573
17	301	298	293
18	191	196	201
19	414	420	427

6. Summary, Conclusion and Future Research Directions

6.1 Summary

The study was centered on the application of MNP model in transportation network analysis. Different aspects of the approach were considered. One of the heuristic link-based algorithms proposed in the literature for solving MNP-SUE problem was investigated. The bias of the method was studied through comparison with the solution of classical MNP-SUE.

Sensitivity analyses were also conducted with respect to different parameters of MNP-SUE problem. In addition, the notion of random-coefficient MNP-SUE was introduced as a new terminology in econometrics area and was formulated for and applied to the route choice problem. While the covariance structure of MNP can address correlation and heteroscedasticity, the mixing procedure is able to represent random taste variation in traveler's route finding.

6.2 Conclusion

The primary findings of the study can be outlined as follows:

Table 6. Mixed MNP-SUE path flow and path time

O-D	Path Number	Link sequence	m=1.0 s=0.2		m=1.0 s=0.4		m=1.0 s=0.6	
			Path Flow	Path Time	Path Flow	Path Time	Path Flow	Path Time
1-2	1	2-18-11	191	43.8	196	43.8	201	43.9
	2	1-5-7-9-11	36	56.4	35	56.3	34	56.1
	3	1-5-7-10-15	26	59.9	26	59.8	25	59.5
	4	1-5-8-14-15	37	58.6	36	58.5	34	58.4
	5	1-6-12-14-15	37	61.2	36	61.1	35	61.1
	6	2-17-7-9-11	23	57.7	22	57.6	22	57.4
	7	2-17-7-10-15	20	61.2	20	61.0	20	60.8
	8	2-17-8-14-15	30	59.9	29	59.8	28	59.7
1-3	9	1-6-13-19	296	54.8	300	54.8	307	54.9
	10	1-5-7-10-16	87	61.8	87	61.6	86	61.3
	11	1-5-8-14-16	104	60.5	104	60.3	103	60.1
	12	1-6-12-14-16	85	63.0	83	62.9	81	62.8
	13	2-17-7-10-16	104	63.0	103	62.8	102	62.6
	14	2-17-8-14-16	124	61.7	123	61.6	121	61.4
4-2	15	4-12-14-15	266	46.6	271	46.6	277	46.6
	16	3-5-7-9-11	145	53.6	144	53.4	141	53.1
	17	3-5-7-10-15	64	57.1	63	56.9	62	56.6
	18	3-5-8-14-15	81	55.8	79	55.6	77	55.4
	19	3-6-12-14-15	44	58.3	43	58.2	43	58.1
4-3	20	4-13-19	95	40.2	97	40.3	100	40.5
	21	4-12-14-16	40	48.5	39	48.4	37	48.4
	22	3-6-13-19	23	51.9	22	51.9	21	52.0
	23	3-5-7-10-16	15	58.9	16	58.6	16	58.3
	24	3-5-8-14-16	18	57.6	18	57.4	18	57.1
	25	3-6-12-14-16	9	60.1	9	60.0	9	59.9

• Results showed a significant level of bias for the studied link-based method in production of MNP-SUE flow.

• Further analysis showed that while the path-based MNP flow is quite sensitive to the value of the calibration parameter of the model appearing in its covariance formulation, the link-based method represents a very low degree of sensitivity.

• Although more efficient due to its link-based nature, the link-based method proved to be significantly dependent on the number of Monte Carlo simulation iterations in prediction of flow while the path-based method approaches to the long-run result quickly. This property makes application of classical probit approach in large-scale networks more promising.

• Investigations also demonstrated that even for very highly congested networks there are still logical reasons for transition from UE to SUE approach.

• Application of random-coefficient approach also illustrated a more degree of randomness in prediction of flow than the classical non-mixed MNP.

6.3 Future Research Directions

• The main concentration of this study was indeed on flow prediction and model estimation problem was in fact beyond the scope the present research. Further studies are required for estimation of structured-covariance MNP models for road network modeling based upon experimental data. Regarding the mixed-MNP modeling, however, it should be noted that owing to the novelty of the approach, to the best of our knowledge, none of the existing choice modeling software packages are able to estimate this model and modelers have to develop their own codes. Further, estimation of mixed MNP models demands far more rich data sets than the classical model.

• Numerical validation of SUE-type models based on flow observations of real-sized networks can still be regarded as a gap in the literature.

• Other link-based probit assignment models proposed in the literature needs to be numerically scrutinized as well to uncover that whether they actually produce MNP-SUE flows.

• To enhance explanatory power of the route choice model, other variables such as path length, fuel price and road tolls can be further introduced to utility functions.

• In addition, to relax the implicit restrictive assumption of fixed travel demand, the STA model can be generalized to a stochastic "mode choice-traffic assignment" model.

• This study focused on theoretical aspects of SUE problem. Further studies, however, needs to be conducted as to the practical aspects of the problem notably choice set generation and efficiency of the equilibrium algorithm. Although the number of previous studies in these areas is rather considerable, but the issues are still open to research.

References

- Akamatsu, T. (1996) "Cyclic flows, Markov process and stochastic traffic assignment", Transportation Research 30B(5), pp.369-386.

- Bekhor S., Toledo,T. and L. Reznikova, L. (2008) "A path-based algorithm for the cross-nested logit stochastic user equilibrium traffic assignment", Computer-Aided Civil and Infrastructure Engineering, Vol 24, pp. 15-25.

- Bekhor, S., Ben-Akiva, M. E. and Ramming, M. S.(2006) "Evaluation of choice set generation algorithms for route choice models", Ann Oper Res, 144, pp. 235-247.

- Bekhor, S., Ben-Akiva,M. S. and Ramming, S. M. (2002) "Adaptation of logitkernel to route choice situation", Transportation Research Record:Journal of the Transportation Research Board, No. 1805, TRB, NationalResearch Council, Washington, D.C., pp. 78–85.

- Bell, M. G. H. (1995) "Alternatives to Dial's logit assignment algorithm", Transportation Research 29B(4), pp.287-296.

- Ben-Akiva, M. S. and Bierlaire, M. (1999) "Discrete choice methods and their applications to short term travel decisions", in: W. Hall (Ed.) "Handbook of Transportation Science", pp. 5–33 (Dordrecht: Kluwer).

- Bovy, P. H. L., Bekhor, S.and Prato,G. (2008) "The factor of revisited path size: Alternative derivation. In Transportation Research Record", Journal of the Transportation Research Board, No. 2076, Transportation Research Board of the National Academies, Washington, D.C., pp. 132–140.

- Burrell, J. E. (1968) "Multiple road assignment and its application to capacity restraint", Proc., 4th. International Symposium on the Theory of Traffic Flow, Karlsruhe, West Germany, pp. 210–219.

- Cascetta, E., Nuzzolo, A., Russo, F. and

Vitetta, A.(1996) “A modified logit route choice model overcoming path overlapping problems: Specification and some calibration results for Interurban Networks”, In Proc., 13th International Symposium on Transportation and Traffic Theory, Pergamon, Lyon, France, pp. 697–711.

- Clark, C. E. (1961) “The greatest of a finite set of random variables”, Operation Research, Vol.9, pp. 145-162.

- Clark, S. D. and Watling,D. P. (2000) “Probit-based sensitivity analysis for general traffic networks”, Transportation Research Record, 1733, pp. 88-95.

- Daganzo, C. F. and Sheffi,Y. (1977) “On stochastic models of traffic assignment”, Transportation Science, Vol. 11, pp. 253–274.

- Dial, R. B. (1971) “A probabilistic multipath traffic assignment algorithm which obviates path enumeration”, Transportation Research, Vol. 5, pp. 83–111.

- Eaton, M. L. (1983) “Multivariate statistics: a vector space approach”, John Wiley and Sons..

- Fisk, C. (1980) “Some developments in equilibrium traffic assignment”, Transportation Research, Vol. 14B, pp. 243–255.

- Florian, M. (1974) “On modeling congestion in dial□s probabilistic assignment model”, Transportation Research, Vol. 8, pp. 85-86.

- Florian, M. and Fox, B. (1976) “On the probabilistic origin of Dial□s multipath traffic assignment model”, Transportation Research, Vol. 10, pp. 339-341.

- Haghani, M., Shahhoseini,Z., Samimi, A. and Aashtiani, A. Z. (2013) “On calibration and application of logit-based stochastic traffic assignment models”, International Journal of Transportation Engineering (IJTE), Vol.1-3, pp. 151-172.

- Horn, R. A. and Johnson, C. R. (1985) “Matrix analysis”, Cambridge: Cambridge University Press.

- Horowitz, J.L., Sparmann, J.M. and. Daganzo, C.F (1982) “An investigation of the accuracy of the Clark approximation for the multinomial probit model”, Transportation Science, Vol. 16(3), pp. 382-401.

- Haghani, M., Shahhoseini, Z. and Sarvi, M. (2014) “Path-based stochastic traffic assignment: an investigation on the effect of choice-sets size, model specification and model calibration on prediction of static flow,93rd Annual Meeting of Transportation Research Board (TRB), Washington, DC.

- Maher, M. (1992) “SAM—A stochastic assignment model”, In Mathematics in Transport Planning and Control (J. D. Griffiths, ed.), Oxford: Oxford University Press, pp. 121–132.

- Maher, M. and Hughes, P. C (1997) “A Probit based stochastic user equilibrium assignment model”, Transportation Research, Vol. 31B, pp. 341–355.

- Nguyen, S. and Dupuis, D. (1984) “An efficient method for computing traffic equilibria in networks with asymmetric transportation costs”, Transportation Science, Vol.18, pp. 185-202.

- Prashker, J. N. and Bekhor, S. (1998) "Investigation of stochastic network loading procedures",Transportation Research Record, 1645, pp. 94–102.

- Prashker, J. N. and Bekhor,S. (2000) "Congestion, stochastic and similarity effects in stochastic user-equilibrium models", In TransportationResearch Record: Journal of the Transportation Research Board,No. 1733, TRB, National Research Council, Washington, D.C., pp. 80–87.

- Prato, C. G. and Bekhor, S. (2006) "Applying branch-and-bound technique to route choice set generation. in transportation research record", Journal of Transportation Research Board, No. 1985, Transportation Research Board of the National Academies, Washington, D.C., pp. 19–28.

- Prato, C. G., and Bekhor, S. (2007) "Modeling route choice behavior: how relevant is the composition of choice set?". In Transportation Research Record, Vol. 2003, pp. 64-73.

- Rice, J. (1995) "Mathematical statistics and data analysis(second ed.)", Duxbury Press.

- Rosa, A. (2003) "Probit based methods in traffic assignment and discrete choice modeling", Ph.D. Dissertation, Napier University.

- Sheffi, Y. and Powell,W.B. (1981) "A comparison of stochastic and deterministic traffic assignment over congested networks", Transportation Research, Vol.15B, pp. 53-64.

- Sheffi, Y. and Powell, W. B. (1982) "An algorithm for the equilibrium assignment problem for random link times", Networks, 12(2), pp. 191-207.

- Train, K. E. (2009) "Discrete choice methods with simulation", Cambridge: Cambridge University Press, New York.

- Von Falkenhausen, H. (1966) "Traffic assignment by a stochastic model", Proceedings, 4th International Conference on Operational Science, pp. 415-421.

- Yai, T., Iwakura, S.and Morichi, S. (1997) "Multinomial probit with structuredcovariance for route choice behavior", Transportation Research,Vol. 31B, pp. 195–207.

- Zhang, K., Mahmassani, H. S. and Lu, C. C. (2008) "Probit-based time-dependent Stochastic user equilibrium traffic assignment model", Transportation Research Record, 2085, pp. 86-94.