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Research Paper

Presenting an Algebraic Method for Optimally Locating Counter Sensors on a Traffic Network for Estimating the OD Matrix

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Abstract

For several decades, finding the optimal location of counting sensors in a traffic network to obtain the best estimates of the O-D matrix has attracted a growing amount of attention. The availability and the accuracy of a priori data in a network such as O-D matrix and route choice probabilities on one hand, and the complexity of the mathematical operations for solving the location problem even in not a large network, on the other hand, are two main concerns of the presented methods. This paper aims to propose a method that identifies optimum locations for counting sensors without utilizing any a priori data. Relying on the network topological characteristics and link travel times as the representation of the network's pattern of trips is the core concept of this study. By taking benefit of the frame theory algebraic operations, needless of any pre-given a priori data, the location set vector with higher coverage on the network route vectors is identified as the optimal location set of the sensor-equipped network links based on its representation in the route-vectors frame. The most probable used paths are identified utilizing an efficient path algorithm. Additionally, by taking advantage of the matrix operations, the novel method obviates the calculations required in methods using linear or non-linear programming solutions. The presented method is applied on a test network and the results show that in comparison to the non-linear programming method, the proposed method finds a better solution.

Keywords: Counting Sensors Location, Frame Theory, O-D Matrix Estimation, Traffic Network

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1. Introduction

1.1. Transportation Network Sensor Location Problem (NSLP)

Traditionally, the origin-destination demand matrices (O-D) which are essential for an extensive variety of analyses in transportation planning and management, are obtained through a time-consuming labor-taking process involving surveys and mathematical modeling. Another type of method commonly used is developing demand models for the estimation of O-D matrices by constructing a direct relation between the O-D flows and some effective variables [1]. From obtaining the data that are generally prone to many gathering errors, to estimating the O-D matrix utilizing the developed models, which are also prone to misspecification errors, the time spent on these activities is technically a matter of concern as well as the heavy budget needed for conducting such a huge duty [2, 3].

For a couple of decades, the utilization of counting sensors in order to obtain the flows of the equipped link has been an accessible tool for many cities used for different purposes like traffic management and traffic light configurations. In a comprehensive review, [4] divided the network sensor location problem (NSLP) approaches mainly into two categories: the sensor location flow-observability problem, and the sensor location flow-estimation problem. Theoretically, if we have a network-specific minimum number of counting sensors well located on a subset of network links, we can mathematically infer all the link flows of the network [5, 6, 7, 8, 9]. This is namely the flow observability problem in which, given a traffic network (N, L) where N and L are the sets of network nodes and links respectively, the flows of the set of unobserved links U (or sometimes route or OD flows) could be inferred from the flows of a minimum set of observed links O by mathematical relations. In these cases, we can calculate flows of U because these flows are related although not independent. In a more recent review, Castillo et al. [10] introduced an integral approach for analysis of the flow observability, estimation, and prediction problems as different optimization problems in terms of their data, variables, constraints, and objective functions.

In this paper, it is assumed that the number of sensors to be located on the network is less than what is required for a flow-observability case, which leads us to a flow-estimation problem. Generally, the sensor location flow estimation problem is an iterative bi-level problem [4] in which, the upper-level deals with identifying the optimum counting locations and the lower level estimates the flows of interest, based on the data obtained from the counts. Although this study only addresses the upper level, i.e. location level, and the O-D matrix estimation is performed by proposed methods in the literature such as non-linear programming, if the number of sensors is sufficient for a full observation, the situation changes. In other words, it is possible that by increasing the number of sensors, there comes a time that the estimation results would somehow be consistent with the flow-observability case.

Taking benefit of the topological aspects of the traffic network, Hu et al. [11], the proposed method to find the minimum subset of counting locations in a network directly from the network topology and the authors showed that for full observability, up to 70% of the network’s links must be observed. Presenting a graphical approach to infer the link flows, He [12] addressed the observability problem by finding the smallest subset of links in a transportation network to be equipped with counting sensors given that no assumptions about prior information are available.

Viti et al. [13] stated that what is sought in an observability problem is to find the tight upper bound of the minimum number of observed flows such that the system becomes fully observable, while at the same time, no redundant information is taken and the observed flows are linearly independent. Hence, the redundancy can be taken as a measure of extra unnecessary traffic counting that might have been eliminated to reach a lower number of counting locations.

Generalizing his “Synergistic Sensor Location” methodology for full flow observability, Ng [14] addressed the partial link flow observability for only a subset of link flows that are known as a priori and are also of interest. Rinaldi and Viti [15] addressed the full and partial observability problem by proposing exact and heuristic solutions for identifying the route sets with as much information as needed for the problem so that adding another route would not add to previous information. Sun et al. [16] defined a health index for each sensor in a network to evaluate the health of the sensor set, thereby filtering the unreliable traffic data obtained from different sensors and eventually, prioritizing the maintenance.

Ng [17] proposed a node-based approach for full observability of network sensor location problems (NSLP). This study aimed to identify the minimum number of counting locations in a network to be instrumented with two kinds of basic and advanced sensors while minimizing the effect of sensor failure on the link flow inference of unobserved links. The proposed method applied the genetic algorithm heuristic to solve the two objective functions of the optimization model. To address the sensor failure and its effects on link flow inference of unobserved links in a full link flow observability problem, Salari et al [18] presented two objective functions, namely, min-max, and min-sum functions to minimize both the maximum probability of not inferring the flow of an unobserved link due to the sensor failure and the maximum effect of a sensor failure on the link flow inference of unobserved links, respectively.

Shao et al. [19] addressed the flow observability problem under steady-state traffic conditions, considering a node-based counting location at the intersections with asymmetric network and the turning ratios as a priori information to design an integer linear programming model for locating the sensors. Bao et al. [20] investigated the credibility of the spatial distribution of traffic information. Through formulating the relationship between the benefit and the number of sensors by credibility functions, they proposed a method for optimizing the numbers and locations of traffic sensors. As the static optimization problem of locating sensors under the steady-state traffic conditions mainly relates to the planning context, real-time data of the sensors are increasingly considered for optimizing network performance during time-dependent variations. Ahmed et al. [21] investigated the accuracy and reliability of traffic state estimates using real-time traffic data from sensors and a CTM simulation model for the prediction of the traffic state.

Castillo et al. [22] argued that using routes as the basis has the advantage of dealing with incidence matrices with zero and one coefficients, resulting in a reduction of required memory (store bits or integer values instead of real numbers), subsequently increasing the calculation speed. In another paper, Castillo et al. [23] specifically addressed the route flow estimation problem considering the minimization of the number of cameras to be used. They also considered scanning errors and error recovery. To address the measurement errors in a complete link flow observability problem, Xu et al. [24] developed a robust network sensor location model by formulating the problem as min-max and min-sum binary integer linear programs.

The flow estimation problems consist of the utilization of methods to find the unobservable flows with the maximum possible precision [10]. This precision is evaluated through statistical or mathematical methods using the estimated flows and prior information that may not be available or accurate enough itself due to measurement errors or time variability in real networks [25]. Differing from flow observation problems, estimation problems mainly take benefit of known O-D flows, namely a priori data which is almost not free of errors, either instrumental or estimation errors. Apparently, these a priori flows are different from true flows that are realistically unknown to us.

The approaches that apply statistical methods for flow estimation from a subset of observed flows include generalized least squares [26], maximum likelihood [27], entropy maximizing [28, 2], Kalman Filter [29, 30, 31], and Bayesian inference method [32].

Another category of approaches belongs to the mathematical programming context such as bi-level programming [33], nonlinear programming [34], linear programming [35]. Liu et al. [36] proposed a formulation containing three 0-1 programming models to determine the maximum number of traffic flows for each sensor location set regarding the time-spatial correlation. The authors designed an ant colony optimization algorithm with a local search procedure to solve the three models.

A comprehensive comparison in terms of the quality of estimating the O-D matrices is presented by [37] for the methods of locating the link flow detectors. The author concluded that with respect to the quality of the O-D matrix estimation, the approaches which maximize the coverage of the O-D pairs have unfavorable results. To evaluate the efficiency of an exact algorithm, a Branch-and-Cut algorithm, and a Clustering Search heuristic to find the optimum number and locations of sensors on a traffic network in order to maximize the number of O-D pairs covered, a comparison study was performed in [38].

The principal concentration of all the existing models is the location level, while the minimization of the estimation errors is not explicitly addressed [4]. Fei and Mahmassani [39] addressed the minimization of the O-D demand uncertainty considering demand coverage in the context of dynamic traffic assignment through a multi-objective bi-level solution to correct a priori O-D demands as well as locating a minimal number of passive point sensors installed in a traffic network.

Combining the various types of data gathered from heterogeneous sensors has been recently considered due to new technologies emerging in ICT and ITS applications. In a recent study, the data from two types of sensors, one measuring the turning ratios at an intersection, and the other measuring the flow in a road are applied to minimize the number of sensors for complete recovery of the vehicular flow [40]. Trying to maximize the information gained by a limited number of counting sensor and AVI readers in an O-D demand estimation problem, a scenario-based stochastic optimization procedure, and a beam search algorithm is proposed by [41] to find suboptimal locations of the point and point-to-point sensors. Hu and Liou [42] developed an integrated heterogeneous sensor deployment model for estimating the O-D demand matrices. Their model takes into account the topological characteristics of the network by adopting the link-node incidence matrix approach, as well as combining the data of passive and active type sensors through separate models to optimally locating the sensors. Continuing this study, in [43], the authors proposed a two-stage optimization model for the deployment of heterogeneous sensors. To determine the O-D matrix, they designed and applied an iterative solution. Fu et al. [44] developed a heterogeneous sensor location model incorporating passive and active sensors to determine the most cost-effective strategies for sensor deployment.

Gentili and Mirchandani [45] reviewed the contributions of optimally locating fixed sensors on a traffic network in terms of estimating travel times. They addressed analytical models of location for both counting sensors and AVI readers. Zhu et al. [46] proposed a link-based method to locate sensors on a network to maximize the travel time information gain while accounting for prior travel time distribution uncertainty.

As it was mentioned, if the number of counting locations is less than what is needed to guarantee network observability, to have the best accurate estimation from the link counts, those links with larger information content, namely the informative links should be selected through a prioritization method. A possible way to identify the more informative location set is to define a location evaluation criterion based on a priori data regarding flows of interest and their estimations obtained from location sets data. Finding the optimum location of counting sites has been repletely addressed in research and among the proposed link selection rules, the 5 most comprehensive locating rules are as follows:

1- O-D covering rule [47], which requires a certain minimum proportion of each O-D pair to be observed in the selected link,
2- Maximal flow fraction Rule [34], in which links with the highest fraction of O-D flow for each O-D pair should be selected,
3- Maximal route covering rule or O-D separation rule [48], that obliges all the routes connecting each OD pair should be covered,
4- Maximal flow interception rule [34], that necessitates the set of links that intercept the maximum number of O-D movements to be selected, and
5- Link independence rule [34], which selects the links with linearly independent flows.

Pando et al. [49] investigated 9 location rules addressed in [50], in terms of the quality of O-D matrix estimation by four different metrics based on the number of sensors.

This paper represents an algebraic method to optimally locate a predefined number of counting sensors on a traffic network by which the optimum estimation of the O-D matrix can be then obtained. In contrary to the most counting location problem methods, which apply a priori data on O-D demands, the proposed method requires no a priori information of the network and mainly relies on topological characteristics and the static travel times of links in the network. The advantage of no necessity for any pre-known O-D flows makes its application plausible for networks without a priori data. The primary idea behind the proposed method is the reduction of the redundant locations which would show some linear dependencies in their link flows (rule 5 above) while increasing the number of network covered routes (rule 3). Network routes are identified by an efficient path identification algorithm and thus no assignment algorithm or prior O-D demand would be necessary.

The rest of the paper is organized as follows. Section 1.2 provides a brief review of algebraic definitions that are used in the proposed method and an extensive literature review regarding the network counting location problem (NCLP). Section 2 describes the methodology of the proposed method and the logic behind it. In section 3, the method is applied on three networks and the results are discussed. Section 4 provides conclusions and future research proposals.

1.2. Mathematical Review

Although algebraic methods have been applied in the past by researchers e.g., [34, 11, 13, 51], this paper addresses the algebraic characteristics of a frame and take benefits of its capabilities to solve the sensor location problem.

A fundamentally crucial concept in vector spaces is the concept of “no linear dependence between the elements of the basis” of a space. In most cases, in addition to the “linear independence” property, the elements are required to have unit length and to be orthogonal to each other (i.e., the orthonormal bases) according to the definition of an inner product [52]. Sometimes, we encounter cases in which there exist some degrees of dependence between the elements based on which we need to construct or reconstruct a vector. “degree of dependence” is equal to the number of repeated elements in the set of basis elements. As an example, the reader may refer to the space defined by three vectors $\varphi_1, \varphi_2, \varphi_3$ depicted in the Figure 1 known as Mercedes Benz Frame [53]. Elimination of this dependence may be impossible or even the dependence may be favorable concerning some aspects. In such cases, a more flexible tool is addressed in the context of the frame theory. A frame for a vector space equipped with an inner product and non-independent elements will allow each vector in the space to be written as a linear combination of the frame elements.

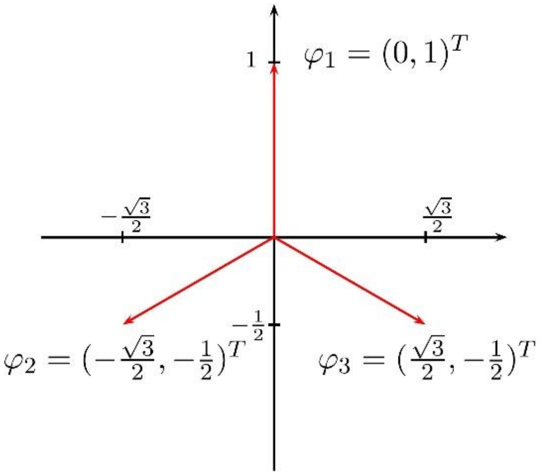

Figure 1. Mercedes Benz Frame *[53]*

Here we define the number of repeated basis elements of an arbitrary vector as the degree of redundancy of the vector. For example, assume a vector v in the first quarter space of the two-dimensional vector space depicted in Figure 1. As it can be seen, vector v when represented with regards to three-dimensional space basis elements φ_1, φ_2, and φ_3, could have two coefficients for the horizontal basis element of the original two-dimensional space (x) and three coefficients for the vertical basis element (y); this is because that in the three frame elements of φ_1, φ_2, and φ_3, there are two repetitions of element x and three repetitions of element y. Measuring this degree for each vector by an appropriate metric can give us special insight into the vectors through comparisons between their redundancy measures. In particular, for a specific instance applicable in this research, if the dependence between the counted flows corresponding to a certain counting location set in a traffic network is assumed not favorable, it may be possible to compare the redundancy degrees of the counting location set vectors in the network in order to find the least dependence-affected counting location set; the very idea that is mainly the foundation stone of this research.

By definition, a countable family of elements $\{f_k\}_{k\epsilon I}$ in V is a frame for V if there exist constants A and B ($0 < A \leq B < \infty$) such that:

$$A\|f\|^2 \leq \sum_{k\epsilon I} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad \forall f \varepsilon V \tag{1}$$

where V is a finite-dimensional vector space, equipped with an inner product [52]. As an example let $(e_i)_{i=1}^{6}$ representing the link vectors of the traffic network in Figure 7, be an orthonormal basis for $\mathcal{H}^6$, where $\mathcal{H}^6$ is a 6-dimensional Hilbert space. Table 3 illustrates a 24-dimensional frame in this space constructed by the vectors represented in its columns, i.e., system $(e_1, e_1, e_1, e_1, e_2, e_2, e_2, e_2, \dots, e_6, e_6, e_6, e_6)$ with vector e_i appearing 4 times, is a frame for $\mathcal{H}^6$ with frame bounds $A = B = 4$. This example is a 4-tight frame.

The constant numbers A and B which are not unique called the frame bounds and they are called the optimum bounds among all others, namely A^* and B^*, when they are the supremum and infimum over all lower and upper bounds, respectively. Every sequence of vectors $\{f_k\}_{k=1}^{m}$ in V is a frame for $Span\{f_k\}_{k=1}^{m}$. Corollary, a family of elements $\{f_k\}_{k=1}^{m}$ in V is a frame if and only if $Span\{f_k\}_{k=1}^{m} = V$. This implies that a frame might contain more elements than needed to be a basis. A frame that is not a basis is called a redundant or over-complete frame.

Now consider a traffic network with n links $l = \{l_i, i = 1,2, \dots, n\}$, each of which represented as a vector containing zeros in all its elements except in the element whose number is identical to the number of the link that takes the number 1, e.g., l_1: $(1,0, \dots ,0)$, l_2: $(0,1, \dots ,0)$, …, l_n: $(0,0, \dots ,0,1)$. The space that is defined by these link vectors is an orthonormal vector space V wherein link vectors are its unit basis elements. Assume that there are m routes in this network with different lengths (different number of constituent links) whose vectors form an m-dimensional frame $r = \{r_i, i = 1,2,\dots,m\}$ for the space V (consider that in traffic networks, the route vector numbers are generally greater than link numbers and they are usually linearly dependent). The synthesis (or pre-frame) operator of this frame is $R_{n\times m}$, which is also a route-link incidence matrix, as follows:

$$R: C^m \to V^n \ , \quad R\{c_k\}_{k=1}^{m} = \sum_{k=1}^{m} c_k r_k \text{ , and} \tag{2}$$

$$R_{n\times m} = \begin{bmatrix} | & | & | & | \\ r_1 & r_2 & \dots & r_m \\ | & | & | & | \end{bmatrix} \tag{3}$$

where C^m is an m-dimensional frame in V. Consider now the set $c = \{c_i, i = 1,2,\dots,p\}$ containing p counting location set each of which represented as an n-dimensional vector in the network space V^n. Thus, the location-set-link incidence matrix would be:

$$C_{n\times p} = \begin{bmatrix} | & | & | & | \\ c_1 & c_2 & \dots & c_p \\ | & | & | & | \end{bmatrix} \tag{4}$$

Since every vector $f \in V$ can be decomposed with respect to the frame operator S (i.e., the vector is analyzed in terms of the frame by computing its frame coefficients) as follows:

$$\begin{aligned} f &= \sum_{k=1}^{m} \langle f, S^{-1} r_k \rangle r_k \\ &= \sum_{k=1}^{m} \langle f, r_k \rangle S^{-1} r_k \end{aligned} \tag{5}$$

where S is obtained by the equation $S = T^*T$, in which T and T^* are analysis and synthesis (adjoint) operator of the frame, respectively, according to the following definition [54]:

$$Tx = \ (\langle x, r_i \rangle)_{i=1}^{m}, \qquad x \in V^n \tag{6}$$

Thus, we can write each counting location set's vector likewise:

$$\begin{aligned} c &= \sum_{k=1}^{m} \langle c, S^{-1} r_k \rangle r_k \\ &= \sum_{k=1}^{m} \langle c, r_k \rangle S^{-1} r_k \end{aligned} \tag{7}$$

By definition, the numbers $\langle c, S^{-1} r_k \rangle, k = 1,2,\dots,m$ are frame coefficients. In other words, by expression $\sum_{k=1}^{m} \langle c, r_k \rangle S^{-1} r_k$ the location set vector c is represented with respect to network route vectors, namely the route frame elements.

Originally in space V, the vectors $c = \{c_i, i = 1,2,\dots,n\}$ have ℓ^2 norms (Euclidian norm) as follows [54]:

$$norm_V(c) = (\sum_{k=1}^{n} \langle c, l_k \rangle^2)^{1/2} \tag{8}$$

where l_k are the basis elements of V. Additionally, they have another norm according to their frame $r = \{r_i, i = 1,2,\dots,m\}$ representations as follows:

$$norm_r(c) = (\sum_{k=1}^{m} \langle c, S^{-1} r_k \rangle^2)^{1/2} \tag{9}$$

As can be seen, in addition to the frame coefficients, the $norm_r(c)$ has the term inverse of the frame operator S^{-1} in its formula and this would result in some changes in the norm of the vectors when their representations are altered from the original orthonormal basis to the frame element representation. These changes are interesting for us and will be discussed later.

Since in above-mentioned formulae S^{-1} is applied, a crucial discussion comes about the invertibility of the matrix S. The advantage of the frame (i.e. redundancy or existence of linear dependence between the frame elements), would entail in non-invertibility property of the frame's synthesis and analysis operators within the principal definition of matrix invertibility property. Albeit, we can obtain a unique Moore-Penrose inverse (or pseudoinverse) matrix for each non-invertible $M \times N$ matrix A, namely A^+, through a singular value decomposition (SVD) process that decomposes the matrix A into three matrices U, Σ, and V^* as follows: [54]

$$A = U\Sigma V^* \Longrightarrow A^+ = V\Sigma^+ U^* \tag{10}$$

Where $U_{M\times M}$ and $V_{N\times N}$ are invertible matrices, Σ^+ is a diagonal matrix arising from the inverse matrix of Σ (i.e. Σ^*) by inverting its nonzero (diagonal) entries. The Moore-Penrose inverse that by definition satisfies four criteria known as Moore-Penrose conditions as follows:

1-$AA^{+}A = A$

2-$A^{+}AA^{+} = A^{+}$

3-$(AA^{+})^{*} = AA^{+}$

4-$(A^{+}A)^{*} = A^{+}A$

finds the minimum Euclidian norm solution to a system of linear equations with multiple solutions. This is the case in our method that we are going to find the coefficients of the location-set vectors with respect to route vectors which generally have infinite solutions.

Principally, the construction of the route vectors' frame requires the identification of the network's routes. As mentioned before, the proposed method does not rely on a priori data specifically on the O-D demand matrix. Nonetheless, it needs the route-link incidence matrix of the network. Utilization of the sensors which help identify the used paths of the network passenger vehicles has been addressed in some studies [55, 56, 57, 45, 58, 59, 60]. Any information about the routes obtained from sources such as vehicle or path ID sensors, Image sensors, RFID tag data, assignment algorithms, interviews, etc., is out of the method's scope. Alternately, we extract the most possible used paths of the network by taking benefit of its topology and link travel times.

There have been many methods proposed for identifying the routes in a traffic network [61] among them the algorithm introduced by [62] is used in this study, however, other algorithms can be used here. It assumes that all trips in the network use efficient paths, nevertheless, obviously a few trip-makers use non-efficient paths that are rendered insignificant. Yang and Zhou [34] applied this algorithm to obtain the turning probabilities of each node without path enumeration or an assignment algorithm, which the latter is the important specification of this algorithm to be used here. Although in future research, the applicability and efficiency of other algorithms can be studied. By definition, the efficient path is the one that takes the trip-maker farther away from her or his origin and closer to the destination via choosing the appropriate next node in the network from the beginning of the trip. The proposed algorithm has the following specifications:

1- Efficient paths have nonzero volumes,
2- Equal-length paths have equal volumes,
3- Longer paths have fewer volumes
4- There are some engineering control parameters for diversion curves,
5- The algorithm eliminates the necessity for path enumeration,

in which the mentioned volumes are route flows. Since the algorithm does not consider the link volumes, apparently the effect of congestion is not accounted for.

The set of all simple or efficient paths in a network can be narrowed by several deterministic or stochastic approaches. Considering the travel time, distance, scenery, congestion, etc., Ben-Akiva et al. [63] developed models to label the probable physical paths in a network in order to identify the optimal path with respect to some criterion functions. These functions are estimated by maximizing the proportion of observed routes in the set of labeled paths. Similar narrowing methods are introduced by [64] applying link penalty, [65] through link elimination, and [66] by a simulation method that produces alternating paths by drawing link costs from probability distributions.

Viti et al. [13] addressed the problem of partial observability based on the pivot procedures primarily introduced in [57, 51] and the Null space of the reduced matrix defining the topology of the network, to present an innovative metric which associates an error to a partially observed network as well as allowing to rank the different full observability solutions. The contribution of this study is introducing a method to locate a predefined number of sensors optimally on the network in terms of minimum redundancy information and maximum route coverage. This method uses the topological characteristics of the network and needs no prior information regarding any kinds of flows. This is important in terms that the prior data are often be unreliable while in this method the optimum locations of the counting sensors are identified independent of the O-D matrix. Based on the coefficients of the location-set vectors with respect to the route vectors frame, a metric is applied as the norm of the location-set vectors in the frame representation. The minimum-normed location-set vector would be the optimum location for the sensors. The counting data obtained from the optimum location set identified by the method is then applied in O-D flow estimation approaches.

2. Methodology

Primarily in this section, we introduce two theorems, which will be then applied in the proposed methodology of finding the optimum set of sensor-equipped links. Firstly, we point out two hints. As by the definition, both the rout-link and location-set-link incidence matrices and in fact, all the route and location set vectors have non-negative elements with values less than or equal to 1 (i.e. either zero or 1). This implies that we deal with a specific part of the n-dimensional space, namely the positive part, in which all the coefficients of the vectors are non-negative. Another point is regarding the number of elements in the location-set-link incidence matrix with values equal to 1. Since the problem of finding the optimum location set deals with a pre-defined number of counting sensors, the number of valued 1 elements for all the location set vectors are identical, and such so the zero-valued ones. Hence, the Euclidian norms of these vectors are all the same and equal $\left(\sum_p (1^2)\right)^{1/2}$, and p is the number of sensors. Now we proceed to theorems.

Theorem 2.1. Assume that the route vectors $\{r_i\}_{i=1}^{m}$are a frame R for a transportation network space V and c_i is an arbitrary counting location set vector. Then,

$$norm_V(c_i) \geq \ norm_R(c_i) \qquad (11)$$

where $norm_V$ and $norm_R$ are the Euclidian norms of the vector c_i with respect to V and R.

Proof. Remind that in our case, i.e. route vectors and location set vectors, all the elements are non-negative and thus we deal with the positive part of the space, i.e. a cone in which every vector has zero or positive coefficients. Without loss of generality, consider the following example in which a 3-dimensional frame with elements r_1, r_2, and r_3 is constructed in a 2-dimensional space with orthonormal bases e_1 and e_2 (Figure 2). As it can be seen, frame elements r_1 and r_2 are the repetition of e_1, while r_3 is identical to e_2.

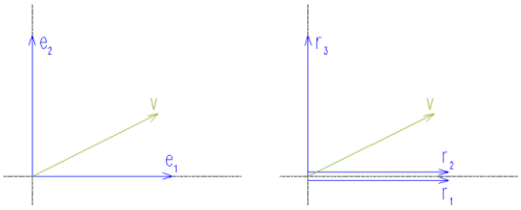

Figure 2. Vectors ($\mathbf{e_1}, \mathbf{e_2}$) representing a 2-D space (left) – Vectors ($\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}$) representing a 3-D frame in the 2-D space (right) and an arbitrary vector v

Assume that vector v is a location set vector in the 2-dimensional space. By adding just one element to the space basis elements namely, r_1, r_2 and r_3, we construct a 3-dimensional frame R in this space with one degree of dependence. The vector v is represented in the space V by its coefficients as follows:

$$v = \sum_{i=1}^{2} a_i e_i \qquad (12)$$

By definition, the Euclidian norm of v is:

$$\left(norm_V(v)\right)^2 = \sum_{i=1}^{2} {a_i}^2 = {a_1}^2 + {a_2}^2 \qquad (13)$$

For the simplicity reasons, due to the positivity of the value of the norms, we use the squared values of the norms.

Remember that $0 \leq a_i \leq 1$ and if so, should the value of a_1 be divided into two other positive parts, then $0 \leq a_{11} \leq a_{12} \leq a_1 \leq 1\ and, a_{11} + a_{12} = a_1$ in order for the vector v to be represented with respect to the frame elements r_1 and r_2. Since there is no redundancy in the second dimension of the space, i.e. there is only one frame element r_3 corresponding the dimension e_2, the coefficient of the vector e_2 will be exactly equal to the coefficient of r_3. Now we calculate the norm of the vector v with regards to the coefficients of the frame, i.e., a_{11}, a_{12} and a_2 as follows:

$$\left(norm_R(v)\right)^2 = {a_{11}}^2 + {a_{12}}^2 + {a_2}^2 \quad (14)$$

By adding the positive value of $2a_{11}a_{12}$ to the norm equation above we have:

$$\left(norm_R(v)\right)^2 + 2a_{11}a_{12} = {a_{11}}^2 + 2a_{11}a_{12} + {a_{12}}^2 + {a_2}^2 \quad (15)$$

The right-hand-side of the above equation is the squared Euclidian norm of v, i.e. ${a_{11}}^2 + 2a_{11}a_{12} + {a_{12}}^2 + {a_2}^2 = (a_{11} + a_{12})^2 + {a_2}^2 = {a_1}^2 + {a_2}^2$ (Equation 12), meaning:

$$norm_V(v) \geq norm_R(v) \quad (16)$$

and this proves Equation (10).

Theorem 2.2. Assume two vectors v_1 and v_2 in frame R with different degrees of redundancy as depicted in Figure 2 (vectors v_1 and v_2 have one and zero degrees of redundancy, respectively). The vector with a higher degree of redundancy is smaller in terms of the frame norm than the vector with a lower degree, i.e., if v_1 has a higher degree of redundancy than v_2, then:

$$norm_R(v_1) \leq norm_R(v_2) \quad (17)$$

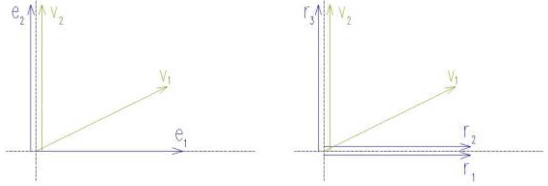

Figure 3. Vectors ($\mathbf{e_1}, \mathbf{e_2}$) representing a 2-D space (left) – Vectors ($\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}$) representing a 3-D frame in the 2-D space (right) and arbitrary vectors $\mathbf{v_1}$and $\mathbf{v_2}$with different degrees of redundancy.

Proof. The proof is straightforward and identical to proposition 3.1.

Based on the theorems above, we devise the proposed method of locating a predefined number of counting sensors on a transportation network as follows. The flow chart of the method's steps is depicted in Figure 6. Theorems 2.1 and 2.2 are important for us in terms of that they imply that between two location set vectors with identical norms in an orthonormal space defined by the traffic network links when they are represented with respect to a frame defined by the route vectors of the network, the one that includes the links with more repeated basis elements (or higher degree of redundancy) will have a smaller norm. In other words, when we choose a location set whose links are included in more network routes, its norm would be smaller than the one with links included in less number of routes than the former. This statement also means that the location set with links that are included in more network routes has more coverage on the network routes.

On the other hand, due to sparsity network coverage of the routes identified by the efficient path algorithm, it is more probable that each link in the set of the optimum location set covers different routes than the others. Stating differently, it is unlikely that a set of routes that are covered by a sensored link would be exactly the same set in the other sensored link. This would result in less dependency between routes that are covered by the optimum location set.

In the methodology of the proposed method, we are going to define a frame in the traffic network space, whose elements are the vectors of the network's routes. The routes are identified by running an efficient paths algorithm based on the steady-state link travel times. However, we do not utilize the constitutional form of the pretold route vectors, but, each vector is decomposed to its constituent basis elements. Then, the vectors of the location sets of the sensors are represented with respect to the route vectors' frame. Considering Theorems 1 and 2, the Euclidian norms of the location set vectors with respect to frame elements would change, which were all initially identical in the original network space. We introduce the minimum-normed location set vector(s) as the optimum set. The proposed method claims that the introduced location set(s), helps in finding the best estimations of the O-D matrix through maximizing the route coverage and minimizing the redundant information.

Primarily the essential matrices for executing the matrix operations are constructed. Subsequently, mathematical calculations are conducted to identify the optimum links needed to be equipped with the specified number of counting sensors.

Assume a transportation network $N(V, A)$ consisting of links a_i, **(**$a_i \in A, i = \{1,2, \dots, n\}$**)**, connecting v nodes, thereof some are origins o_i, ($o_i \in O$ & $O \subseteq V$), some destinations d_i, ($d_i \in D$ & $D \subseteq V$), and the others are intermediate nodes. Firstly, We assign a travel time value of a non-congested state to each link of the network. Then, a shortest-path algorithm is used to determine the shortest paths between all nodes of the network. In this process for each node, two sets of numbers indicating the shortest-path time are prepared. The first set includes the time from all the origins to that node, and the second set is the time from that node to all the destination nodes.

Subsequently, the efficient path algorithm introduced by [62] is run on the network resulting in the identification of the efficient path vectors r_i, ($r_i \in R, i = \{1,2, \dots, m\}$). Inherently, each link in every identified efficient path has its starting node closer than its ending node to the origin node and contrarily farther to the destination node. Then, the initial route-link incidence matrix is constructed by inserting the efficient path vectors in its columns. Each column element takes number 1 representing the existence of the link in the route, or zero otherwise (Figure 4a).

$$R_{n\times m}: \begin{array}{c} \\ a_1 \\ a_2 \\ \cdots \\ a_n \end{array} \begin{array}{c} \begin{array}{cccc} r_1 & r_2 & \cdots & r_m \end{array} \\ \begin{bmatrix} 1 & \dots & \dots & 0 \\ 1 & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 \\ 0 & \dots & \dots & 1 \end{bmatrix} \end{array}$$

Figure 4a. Initial efficient-paths (route) link-incidence matrix

Afterward, we obtain a matrix introduced here consisting of link elements of the network routes in its columns which for brevity is called route matrix. To do so, we decompose the initial route-link incidence matrix to a certain number of network basis vectors by deconstructing each route vector to its constituent link vectors and replacing the original route vector with these constituent elements in several columns of the route matrix. In other words, each route vector is a linear combination of several link vectors all with the coefficients 1. We deconstruct each route vector by dividing all these link vectors for each route vector and then by placing them in the columns of the network route vectors matrix, we would have a matrix in which many columns are repeated several times, hence resulting in redundancy property of the matrix (Figure 4b).

$$R_{n\times x}: \begin{array}{c} \\ a_1 \\ a_2 \\ \cdots \\ a_n \end{array} \begin{array}{c} \begin{array}{cccc} \overbrace{\qquad\quad}^{r_1} & r_2 & \cdots & \overbrace{\qquad\quad}^{r_m} \end{array} \\ \begin{bmatrix} 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 1 & \dots & \dots & 0 & 0 \\ 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{bmatrix} \end{array}$$

Figure 4b. Decomposed efficient-paths (route) link-incidence matrix

In the next step, as depicted in Figure 5 we find the matrix "S" containing s sensor location set vectors as the combination of p counting sensors on "n" links in the network, i.e., $s = \binom{n}{p}$. The sensor location sets link-incidence matrix is then build up from these vectors positioned in the columns of the matrix. Each column element takes number 1 representing the installation of a sensor on that link, or zero otherwise. As it is mentioned before, the column vectors all have an equal number of ones and zeros in their rows. This would entail an equal Euclidian norm of all the vectors. The number of ones represents the number of counting sensors that mend to be installed on the network links.

$$S_{n\times s}: \left[\begin{array}{c|cccc} & s_1 & s_2 & \cdots & s_s \\ \hline a_1 & 1 & \cdots & \cdots & 0 \\ a_2 & 1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 \\ a_n & 0 & \cdots & \cdots & 1 \end{array}\right]$$

Figure 5. A sensor location set matrix

Finally, by multiplication of the sensor location set matrix by Moor-Penrose pseudo-inverse of the route matrix, another matrix is obtained with the column elements representing the expansion coefficients of each sensor location set with respect to the route vectors frame.

By taking benefit of the property of vector norm changing described by theorem 2.2, the location vector set(s) with the minimum norm is drawn out of the whole set as the optimum location of the sensors on the network. In the Figure 6, the flowchart of the proposed method is illustrated.

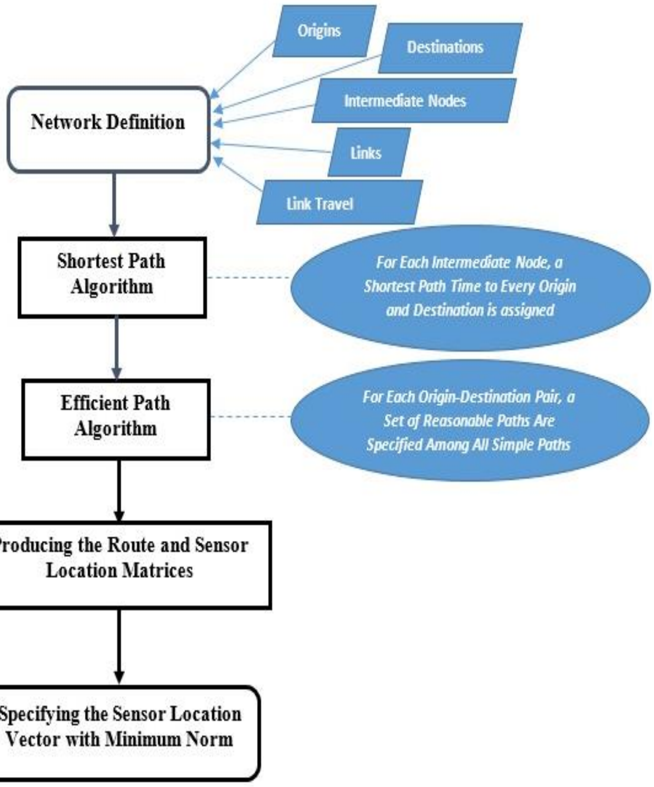

Figure 6. Flowchart of the proposed method, highlighted parallelograms show the network specification inputs, the ellipses are explanations and the rectangles show the process of the method from start to the end

3. Numerical Analysis

In this section, we analyze three different test networks by applying the proposed frame theory method to identify the optimum location set of sensors and discuss the results. The first network is a simple 6-node and 6-link network adopted from [4] with 3 origins and 3 destinations. The second example network is a fishbone network applied by [11] and the last one, is Sioux Falls' city network. It should be mentioned that due to the large number of efficient paths generated by Dial's algorithm, there are severe computational limitations for running the method on larger networks. These limitations also deter the authors from running the method on the Sioux Falls network with full origin-destination nodes.

3.1. The 6-node and 6-link network

Consider the network in Figure 7 primarily introduced by [47] as an example. As stated in [4] assume that there are 8 O-D pairs in this network as $w_1 = (1,5)$, $w_2 = (1,6)$, $w_3 = (1,4)$, $w_4 = (2,4)$, $w_5 = (2,5)$, $w_6 = (2,6)$, $w_7 = (3,5)$, and $w_8 = (3,6)$.

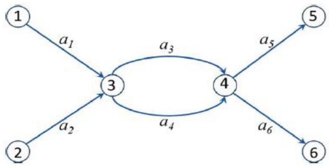

Figure 7. A network consisting of 6 nodes and 6 links [4]

In Table 1, these O-D pairs and corresponding routes, which are identified by a method other than Dial's algorithm, are illustrated and Table 2 shows the link travel times in minutes.

Table 1. O-D pairs and corresponding routes for the network of Figure 7

O-D pair	True O-D	Routes	Links		
$\boldsymbol{w_1} = (\mathbf{1}, \mathbf{5})$	50	R_1	a_1	a_3	a_5
		R_2	a_1	a_4	a_5
$\boldsymbol{w_2} = (\mathbf{1}, \mathbf{6})$	100	R_3	a_1	a_3	a_6
		R_4	a_1	a_4	a_6
$\boldsymbol{w_3} = (\mathbf{1}, \mathbf{4})$	20	R_5	a_1	a_3	
		R_6	a_1	a_4	
$\boldsymbol{w_4} = (\mathbf{2}, \mathbf{4})$	200	R_7	a_2	a_3	
		R_8	a_2	a_4	
$\boldsymbol{w_5} = (\mathbf{2}, \mathbf{5})$	20	R_9	a_2	a_3	a_5
		R_{10}	a_2	a_4	a_5
$\boldsymbol{w_6} = (\mathbf{2}, \mathbf{6})$	120	R_{11}	a_2	a_3	a_6
		R_{12}	a_2	a_4	a_6
$\boldsymbol{w_7} = (\mathbf{3}, \mathbf{5})$	100	R_{13}	a_3	a_5	
		R_{14}	a_4	a_5	
$\boldsymbol{w_8} = (\mathbf{3}, \mathbf{6})$	50	R_{15}	a_3	a_6	
		R_{16}	a_4	a_6	

Table 2. Link travel time of the network in Figure 7

Link	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{a_6}$
Travel Time	6	8	4	5	5	8

Table 3. The route vector matrix (for only 8 paths out of 16) of the test network in Figure 7

	Path$_1$			Path$_2$			Path$_3$			Path$_4$			Path$_5$			Path$_6$			Path$_7$			Path$_8$		
a_1	1	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_2	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0
a_3	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0
a_4	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0
a_5	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0
a_6	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1

Table 4. All the location sets considering two sensors in the 6-link Network of Figure 7

	L.S. 1	**L.S. 2**	**L.S. 3**	**L.S. 4**	**L.S. 5**	**L.S. 6**	**L.S. 7**	**L.S. 8**	**L.S. 9**	**L.S. 10**	**L.S. 11**	**L.S. 12**	**L.S. 13**	**L.S. 14**	**L.S. 15**
$\boldsymbol{a_1}$	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
$\boldsymbol{a_2}$	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0
$\boldsymbol{a_3}$	0	1	0	0	0	1	0	0	0	1	1	1	0	0	0
$\boldsymbol{a_4}$	0	0	1	0	0	0	1	0	0	1	0	0	1	1	0
$\boldsymbol{a_5}$	0	0	0	1	0	0	0	1	0	0	1	0	1	0	1
$\boldsymbol{a_6}$	0	0	0	0	1	0	0	0	1	0	0	1	0	1	1

Table 5. The squared values of the norms of the location sets for the network in Figure 7

Location Set	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Norm	0.33	0.28	0.28	0.33	0.33	0.28	0.28	0.33	0.33	0.11	0.28	0.28	0.28	0.28	0.33

Table 6. Comparing True and estimated OD flows of the network on Figure 7

w_i = OD Pair	w_1=(1,5)	w_2=(1,6)	w_3=(1,4)	w_4=(2,4)	w_5=(2,5)	w_6=(2,6)	w_7=(3,5)	w_8=(3,6)
$\widehat{h}_{w_i}$= Estimated OD Flows	50	100	20	200	20	120	100	50
h_{w_i}= True OD Flows	50	100	20	200	20	120	100	50
Difference	0	0	0	0	0	0	0	0

Table 7. The estimated OD flows from optimum location sets in (Gentili & Mirchandani, 2012)

	$Location$ Set	w_1=(1,5)	w_2=(1,6)	w_3=(1,4)	w_4=(2,4)	w_5=(2,5)	w_6=(2,6)	w_7=(3,5)	w_8=(3,6)
Estimated OD Flows	S4: (a1,a5)	55	97.5	17.5	190	17.5	110	97.5	40
	S5: (a1,a6)	47.5	105	17.5	190	10	117.5	90	47.5
	S8: (a2,a5)	47.5	90	10	197.5	25	117.5	97.5	40
	S9: (a2,a6)	40	97.5	10	197.5	17.5	125	90	47.5
	True OD Flows	50	100	20	200	20	120	100	50

Due to the simplicity of the network of Figure 7, the result of running the efficient path algorithm on this network shows that the efficient and simple paths are the same and identical to Table 1. Consequently, the route vector matrix of this sample network based on the efficient paths identified by Dial's algorithm is obtained as is illustrated partly in Table 3. Note that each of the illustrated paths is decomposed to their constituent basis elements, which for these 8 paths there are three basis elements for each route.

Since we are going to install two counting sensors on this network, the number of location set vectors will be $C(n,r) = \binom{6}{2} = 15$, where n is the number of links and r, the number of sensors. These location sets are shown in Table 4.

By multiplication of Moore-Penrose pseudo-inverse of route matrix by sensor location sets matrix we will obtain the expansion coefficients of the location set vectors with respect to the route vectors frame. This is importantly worth pointing out that since all the vectors contributing to the above matrix operations are non-negative and the Moore-Penrose pseudo-inverse finds the minimum normed inverses for the sensor location sets with respect to the route vectors frame, these expansion coefficients will all be non-negative. The Euclidian norms of the obtained vectors are displayed in Table 5.

As it can be seen, the location set no. 10 has the minimum value of the squared vector norm. This implies that, based on the characteristics of this norm which previously discussed, if the links a_3 and a_4 are equipped with counting sensors, the maximum number of the routes are intercepted with minimum dependencies between them. By investigating the 16 routes of the network, it is realized that links a_3 and a_4 each intercept 8 routes which is the maximum in comparison to other links.

Thus, the optimum location set identified by the proposed method for this network is location set no. 10 recommending links 3 and 4 to be equipped with counting sensors. Considering the rule of maximum route covering, this result would have been deemed apparent by taking a glance at the network and investigating its routes.

Referring to table 8 in [4], the true OD flows are extracted and the results of non-linear programming calculations that are carried out to estimate the OD flows for location set no. 10, are compared with the true OD flows as follows in table 6.

As it can be seen, when location set no. 10 is chosen as the sensor-equipped links of the network, based on its counts, the estimated OD flows are exactly equal to the true OD flows. So it can be concluded that this location set is the best optimum due to true O-D flows. However, the true O-D flows are not known to us, we expect that the selected location set will produce the estimates more closely to the unknown reality.

We use the true OD flows instead of a priori flows, since as it was mentioned, it is assumed that the topology of a network somehow geometrically represents its true OD flows. In the real world, these flows are not known, nevertheless, we take benefit of the topological characteristics of the network as the latent variable for its true OD flows.

According to [4], the optimum is obtained as location sets 4, 5, 8, and 9 with the optimum value z_i^* equaling 450 and the OD flows calculated as shown in the Table 7.

As it can be seen from table 5, the recognized optimum location sets in [4], calculates the O-D flows with bigger differences with the true O-D flows than the locations set recognized by the proposed method as the optimum one. However, we should not lose the side of the fact that the former calculations are based on a priori O-D flows and this would result in different solutions when different prior information is applied. But, the method proposed by this study solely relies on the topology and link travel times of the networks and this property distinguishes the method from the others in terms of that it only has one solution (which of course could be a multiple location sets solution) that almost can be attributed to the true O-D flows which influence the network, topologically and traffically.

3.2. The Fishbone network

The second network which is taken from [11] consists of 10 nodes, 18 links, and 4 O-D pairs, i.e. 1 and 2 as origins, and 9 and 10 as destinations (Figure 8).

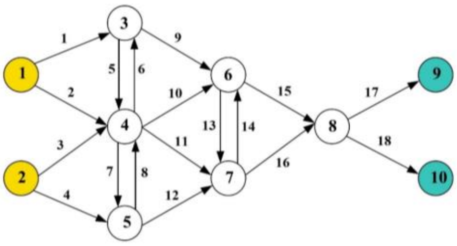

Figure 8. The fishbone network with the link and node numbers

Table 8 demonstrates the link numbers and travel times of this network. Although the network is geometrically symmetric, the travel times of the links are not so. This may influence the symmetry of the paths obtained by the efficient paths algorithm.

Table 8. Link numbers and travel times of the network of Figure 8

Link Number	Starting Node	Ending Node	Travel Time
1	1	3	3
2	1	4	5
3	2	4	5
4	2	5	4
5	3	4	4
6	4	3	4
7	4	5	5
8	5	4	3
9	3	6	5
10	4	6	3
11	4	7	5
12	5	7	5
13	6	7	5
14	7	6	4
15	6	8	3
16	7	8	2
17	8	9	5
18	8	10	4

In the first experiment, we take the number of counting sensors equal to 2 and the result of executing the method shows that the optimum locations for this network are the sets:

The optimum location set:

$\{\{15,16\},\{15,17\},\{15,18\},\{16,17\},\{16,18\},\{17,18\}\}$

Intuitively, due to the simplicity and symmetry of the network, the location sets $\{15,16\}$ and $\{17,18\}$ were somehow predictable. However, the others have been identified by a deeper investigation of the routes. It means that the method shows that there are other location sets with the same coverage on the network routes. This is another valuable property of the proposed method that in some situations, it provides the decision-makers with a choice of several location sets to install a predefined number of counting sensors on the locations that may have other analytical applications and management capabilities as well. The total number of location sets are 153 (i.e. $\binom{18}{2}$), and the size of the route-link incidence matrix is 18×336.

If we locate 4 sensors on this network, the method indicates that the sole optimum location sets would be: $\{15,16,17,18\}$. In this case, The total number of location sets are 3060 (i.e. $\binom{18}{4}$), and the size of the route-link incidence matrix is similar to the previous case, i.e. 18×336.

The method indicates a second-best solution with a higher norm of the location set vectors for locating the 4 sensors as the sets:

The second-best optimum location set:

$\{\{13,15,16,17\},\{13,15,16,18\},\{13,15,17,18\},\{13,16,17,18\},$
$\{14,15,16,17\},\{14,15,16,18\},\{14,15,17,18\},\{14,16,17,18\},$
$\{12,15,16,17\},\{12,15,16,18\},\{12,15,17,18\},\{12,16,17,18\},$
$\{11,15,16,17\},\{11,15,16,18\},\{11,15,17,18\},\{11,16,17,18\},$
$\{10,15,16,17\},\{10,15,16,18\},\{10,15,17,18],\{10,16,17,18\},$
$\{9,15,16,17\},\{9,15,16,18\},\{9,15,17,18],\{9,16,17,18\}\}$

By investigating these second-best solution sets, which have a vector norm higher than the optimum one by 13%, what seems interesting is that since the shape of this fishbone network has a narrow shape at its tail and wider in the middle and the head, thus the links in the vicinity of the tail intercept more routes. This concept is in accordance with the results obtained by executing the proposed method.

3.3. The Sioux Falls network

The city of Sioux Falls network contains 24 nodes and 74 two-way links (Figure 10). Due to computational limitations, only 4 origins and 4 destinations of this network namely, (1,2,7,18) and (20,21,24,13) respectively, are determined for identifying the optimum location set for locating two counting sensors on this network.

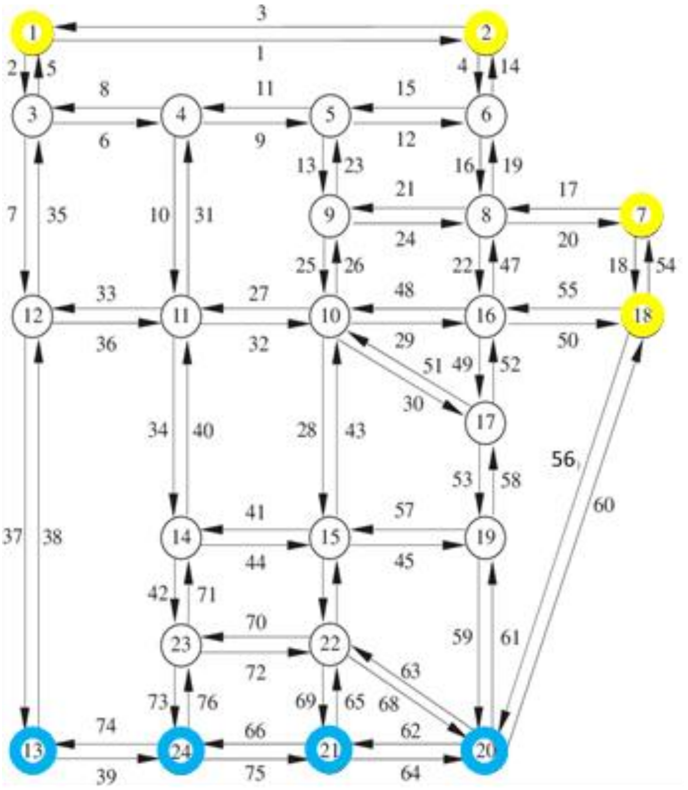

Figure 9. The Sioux Falls network with 4 origins and destinations in yellow and cyan, respectively

The results of executing Dial’s algorithm identifies 12102 efficient paths connecting these origins and destinations. To locate two sensors on this network there are 2850 combinations of locations among which, the method recognizes the following location sets as the optimum sets ranked from 1 to 10:

- The first optimum location set: $\{4,24\}$
- The second optimum location set: $\{2,9\}$
- The third optimum location set: $\{2,13\}$

- The fourth optimum location set: {2,24}
- The fifth optimum location set: {4,22}
- The sixth optimum location set: {3,18}
- The seventh optimum location set: {3,56}
- The eighth optimum location set: {7,53}
- The ninth optimum location set: {16,53}
- The tenth optimum location set: {3,37}

The above-ranked location sets differ from each other by much less than 1% of the norm. Actually, the tenth optimum location set has a norm almost 3.4% larger than the first one. In such situations in which there is the best location set due to calculations of the norms, while, several worse location sets have a very small difference in the size of the norms, we can define a range of tolerance by which, a set of location sets with the different but close size of norms are recognized as the set of optimum location sets. Since in the upper level of NSLP, i.e. O-D flow estimation, there are also estimation errors, this negligence of the differences in the norm size, may be unimportant and have an insignificant effect on the estimations.

Figure 10, graphically shows the amount of repetition of the network links in the 12102 efficient paths. The darker red the link is, the more it appears in the network paths. Logically we expect that the optimum location sets identified by the proposed method be consistent with their color saturation in Figure 10. This can be investigated from this figure, considering the links that are numbered which are present in the above-mentioned optimum sets.

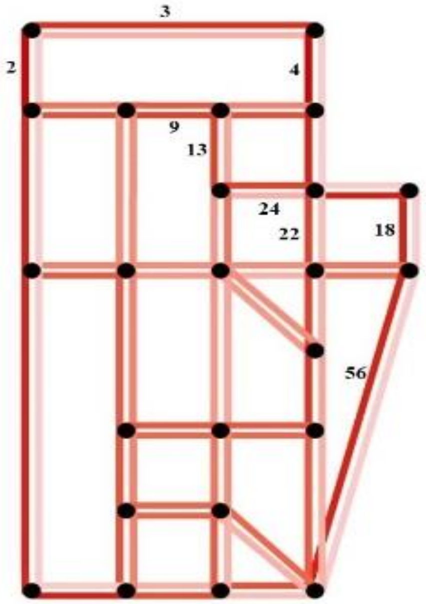

Figure 10. Sioux Falls network and the links repetitions in the efficient paths

An important discussion that may arise is about the justifications for the utilization of an efficient path algorithm that would generate a large number of paths which may undermine the applicability of the method. Logically, we assume that the shortest paths in terms of ravel time can be efficient too. However, in absence of any prior information that helps to identify the most probable used paths of a network, one practical tool, at least in small or medium-size networks is the approaches that take benefit of the topological characteristics and link travel times of the network. Additionally, this assumption is not a limitation to the proposed method and if there is another approach that identifies the network routes more efficiently, its results can be applied as an input to the proposed method.

On the other hand, assume a city whose authorities would like to equip the traffic network with a set of counting sensors that can help to analyze the link flows data for management purposes along with a prospect of completing the set of counting sensors for optimum estimating or updating the O-D flows of the city. In such case, the proposed method is appropriately applicable.

Although more applicable approaches and technological tools for network route identification are being introduced, and ICT tools are evolving fast which could facilitate more complicated computations, it should be considered that eventually, the proposed method would require a priori information to optimally estimate the O-D matrix in the lower level of NSLP. In other words, the most important contribution of this method is that it obviates the repetition of a non-linear programming optimization problem for each location set in the network, as in [4], and leaves the optimization procedure with only one set of optimum counting locations for estimation problem.

An interesting application of the proposed method can be in large-scale networks with a large number of counting sensors installed on a set of network links. The method can replace its efficient path identification procedures with the pre-identified routes obtained through an assignment procedure reducing the massive calculations and resources related to the efficient path algorithm, and on the other hand, choose among the installed sensors those with most information and least dependence, resulting in reducing the input data, to construct the required path and location set incidence matrices in order to optimally estimate the O-D matrices. These two modifications to the method would substantially reduce its computational burden and thanks to the existence of an efficient ICT infrastructure, the method could be applied in large-scale networks for real-time updating the O-D matrices.

It should be mentioned that solving non-linear programming methods by personal computers for NSLP is not often possible within an acceptable running time. However, the running time of the proposed method for solving the optimum location problem of the Sioux Falls network with an Acer TravelMate 8481G laptop equipped with an Intel® Core™ i7, 2.637M, 1.7GH and 6 GB DDR3 memory takes 26 minutes and 25 seconds to obtain the solution.

4. Conclusions

In this study, an algebraic method in the context of frame theory is proposed to solve the network sensor location problem in order to find the optimum location set of a specified number of counting sensors on a transportation network to be then utilized in estimating the OD flows. The proposed method takes advantage of the network topology by applying the link-incidence matrices for network routes and location sets to look up the optimum vector of sensor location set(s) through matrix calculations. The used routes are identified through an efficient path algorithm that disencumbers the method from any assignment algorithms and thereby a priori O-D matrix. In parallel, it releases the solution process from assumptions of prior knowledge about the O-D flows, user route-choice behavior, or intersection turning ratios.

With the assumption that no prior information of the network flows or routes is available for identifying the optimum location of counting sensors, we use the algorithms which identify the efficient routes of the network. These algorithms require complex computations and resources which may seem unrealistic in large-scale networks. However, further research may provide more efficient algorithms that can be developed, while more efficient computational capabilities are growingly available. Although in many small or middle-size networks, the existing algorithms are still applicable. Note that in absence of a priori data, these algorithms are almost the only tools we have. In many cases with the availability of prior information, the proposed method can also act more efficiently which makes it more suitable for both large-scale networks, and real-time updating of the O-D matrices.

By applying the method on three test networks, the results show that the identified location set can provide better estimates of the O-D matrix in comparison to the non-linear programming method, which necessitates multiple iterations of solving the problem for all the possible location sets of the network. The novel method explicitly points out the location set which has better coverage on the network routes and better eliminates the redundant-information locations. On the other hand, since the proposed method does not rely on a priori data, its solution(s) would not change by time-dependent changes in such prior information. Rather, its solution(s) has (have) more tendency to the unknown true O-D flows which are the main influential factor of the topological characteristics of the network.

The proposed method is applicable in networks without any pre-installed sensors or prior information about the O-D flows as well as those which have been equipped with these instruments probably mainly with the purpose of management activities. In the latter case, we would have some pre-fixed counter sensors that may be non-optimally located for flow estimation purposes, and a location problem to be solved for the new set of sensors that should complement the previous set in terms of optimality for O-D flow estimation purposes. The latter case could be considered for further research.

The study may in many instances come to multiple solutions due to network topological symmetries, network route similarities, or link travel times influences. This property provides flexibility in instrumenting different links by choosing different location sets while obtaining the same estimation results. This variation could be useful when considering several importance or cost criteria in choosing the links to be equipped with sensors. Thus, in many network cases, this potential is available to make use of the rankings provided by road and traffic agencies while satisfying multiple considerations.

Application of algorithms more efficient than Dial’s for network path identification can be proposed for further research as well as other technological tools which help identifying the routes of a network such as drone surveillance. This is very crucial to have a faster algorithm that requires less computational resources especially for large-scale networks and also real-time state estimations. Also, data fusion of heterogeneous sensors in the context of frame theory is an open area of continuing research for this study. If the efficiency of the proposed method in terms of analysis time and memory resources for large-scale networks is proven, another interesting area of research would be the application of the proposed method in the dynamic traffic assignments in such networks for real-time O-D matrix updating.

5. References

– E. Cascetta, Transportation_Systems_Analysis: Models and Applications, New York: Springer, 2009.

– H. J. Van Zuylen and L. G. Willumsen, "The Most Likely Trip Matrix Estimated from Traffic Counts," Transportation Research Part B: Vol. 14B, pp. 281-293, 1980.

– W. H. K. Lam and H. P. Lo, "Estimation of origin-destination matrix from traffic counts: a comparison of entropy maximizing and information minimizing models," Transportation Planning and Technology, Vol. 16, pp. 85-104, 1991.

– M. Gentili and P. Mirchandani, "Locating sensors on traffic networks- Models, challenges and research opportunities," Transportation Research Part C 24, p. 227–255, 2012.

– L. Bianco, G. Confessore and P. Reverberi, "A Network Based Model for Traffic Sensor Location with Implications on O/D Matrix Estimates," Transportation Science, pp. Vol. 35, No. 1, pp. 50–60, 2001.

– L. Bianco, G. Confessore and M. Gentili, "Combinatorial aspects of the sensor location problem," Annals of Operations Research 144, p. 201–234, 2006.

– D. Morrison and S. Martonosi, "Where are all the cars? Characteristics of optimal solutions to the sensor location problem," INFORMS Annual Meeting 2008, Washington DC, 2008.

– D. Morrison, S. Martosoni and K. Tucker, "Characteristics of Optimal Solutions to the Sensor Location Problem.," Thesis, Harvey Mudd College, Department of Mathematics, 2008.

– L. Bianco, R. Cerulli, C. Cerrone and M. Gentili, "An optimization mathematical model and an exact algorithm for the sensor location problem," In: Proceedings of Euro Working Group in Location Analysis, 2010.

– E. Castillo, Z. Grande, A. Calviño, W. Y. Szeto and H. K. Lo, "A State-of-the-Art Review of the Sensor Location, Flow Observability, Estimation, and Prediction Problems in Traffic Networks," Hindawi Journal of Sensors, 2015.

– S. Hu, S. Peeta and C. Chu, "Identification of vehicle sensor locations for link-based network traffic applications," Transportation Research Part B 43 (8–9), p. 873–894, 2009.

– S.-x. He, "A graphical approach to identify sensor locations for link flow inference," Transportation Research Part B, p. 65–76, 2013.

– F. Viti, M. Rinaldi, F. Corman and C. M. Tampère, "Assessing partial observability in network sensor location Problems," Transportation Research Part B 70 (2014), p. 65–89, 2014.

– M. Ng, "Partial link flow observability in the presence of initial sensors: Solution without path enumeration," Transportation Research Part E, 51, p. 62–66, 2013.

– M. Rinaldi and F. Viti, "Exact and approximate route set generation for resilient partial observability in sensor location problems," Transportation Research Part B 105, p. 86–119, 2017.

– Z. Sun, W.-L. Jin and M. Ng, "Network sensor health problem," Transportation Research Part C 68, pp. 300-310, 2016.

– M. Ng, "Synergistic sensor location for link flow inference without path enumeration: a node-based approach," Transportation Research Part B: Methodological 46 (3), p. 781–788, 2012.

– M. Salari, L. Kattan, W. H. Lam and H. Lo, "Optimization of traffic sensor location for complete link flow observability in traffic network considering sensor failure," Transportation Research Part B 121, p. 216–251, 2019.

– M. Shao, L. Sun and X. Shao, "Sensor location problem for network traffic flow derivation based on turning ratios at intersection," Mathematical Problems in Engineering, 2016.

– X. Bao, H. Li, L. Qin, D. Xu, B. Ran and J. Rong, "Sensor Location Problem Optimization for Traffic Network with Different Spatial Distributions of Traffic Information," Sensors, 2016.

– Ahmed, D. Watling and D. Ngoduy, "Significance of sensor location in real-time traffic state estimation," in Fourth International Symposium on Infrastructure Engineering in Developing Countries, IEDC, Karachi, Pakistan, 2013.

– E. Castillo, I. Gallego, S. Sánchez-Cambronero and A. Rivas, "Matrix Tools for General Observability Analysis in Traffic

Networks," IEEE Transactions on Intelligent Tranportation Systems, Vol. 11, No. 4, pp. 799-812, 2010a.

– E. Castillo, I. Gallego, J. M. Menéndez and A. Rivas, "Optimal Use of Plate-Scanning Resources for Route Flow Estimation in Traffic Networks," IEEE TRANSACTIONS ON INTELLIGENT TRANSPORTATION SYSTEMS, VOL. 11, NO. 2,, 2010b.

– X. Xu, K. L. Hong, A. Chen and E. Castillo, "Robust network sensor location for complete link flow observability under uncertainty," Transportation Research Part B 88, pp. 1-20, 2016.

– N. Wang and P. Mirchandani, "Sensor location model to optimize origin-destination estimation with a bayesian statistical procedure," Transportation Research Record, pp. No. 2334, pp. 29–39., 2013.

– E. Cascetta, "Estimation of trip matrices from traffic counts and survey data: a generalized least squares estimator," Transportation Research Part B 18 (4–5), p. 289–, 1984.

– H. Spiess, "A maximum likelihood model for estimating origin–destination matrices," Transportation Research Part B 21 (5), p. 395–412, 1987.

– D. Teodorovic, M. van Aerde, F. Zhu and F. Dion, "Genetic algorithms approach for locating automatic vehicle identification equipment locations," Journal of Advanced Transportation 36 (1), p. 1–21, 2002.

– N. Nihan and G. Davis, "Recursive estimation of origin–destination matrices from input/output counts," Transp. Res. Part B 21 (2), p. 149–163, 1987.

– X. Fei, S. Eisenman and H. Mahmassani, "Sensor coverage and location for real-time traffic prediction in large-scale networks," In: Proceedings of the 86th Annual Meeting of the Transportation Research Board, Washington DC, 2007.

– Fisk, "On combining maximum entropy trip matrix estimation with user optimal assignmwnt," Transportation Research B, pp. Vol. 22, pp. 69–79, 1988.

– J. Maher, "Inferences on trip matrices from observations on link volumes: a Bayesian statistical approach," Transp. Res. Part B 17 (6), p. 435–447, 1983.

– H. Yang, T. Sasaki, Y. Iida and Y. Asakura, "Estimation of origin–destination matrices from link traffic counts on congested networks," Transp. Res. Part B26, p. 417–434, 1992.

– H. Yang and J. Zhou, "Optimal traffic counting locations for origin–destination matrix estimation," Transportation Research Part B 32, p. 109–126, 1998.

– H. Sherali, R. Sivanandan and A. Hobeika, "A linear programming approach for synthesizing origin–destination trip tables from link traffic volumes," Transp. Res. Part B 28 (3), p. 213–233, 1994.

– Y. Liu, N. Zhu, S. Ma and N. Jia, "Traffic sensor location approach for flow inference," IET Intelligent Transport Systems, 2014.

– T. Larsson, "Allocation of Link Flow Detectors for Origin-Destination Matrix Estimation— A Comparative Study," Computer-Aided Civil and Infrastructure Engineering 25, p. 116–131, 2010.

– V. H. Souza de Abreu, P. H. Gonz´alez, G. R. Mauri, G. M. Ribeiro, R. D. Orrico, N. F. R.

Campos Júnior and C. A. Abramides, "Network sensor location problem with monitored lanes: Branch-and-cut and clustering search solution techniques," Computers & Industrial Engineering 150, 2020.

– X. Fei and H. S. Mahmassani, "Structural analysis of near-optimal sensor locations for a stochastic," Transportation Research Part C 19, pp. 440-453, 2011.

– M. Rodriguez-Vega, C. C.-d.-W. Canudas-de-Wit and H. Fourati, "Location of turning ratio and flow sensors for flow reconstruction in large traffic networks," Transportation Research Part B 121, pp. 21-40, 2019.

– X. Zhou and G. F. List, "An Information-Theoretic Sensor Location Model for Traffic Origin-Destination Demand Estimation Applications," Transportation Science, Vol. 44, No. 2, p. 254–273, 2010.

– S.-R. Hu and H.-T. Liou, "A generalized sensor location model for the estimation of network origin–destination matrices," Transportation Research Part C 40, pp. 93-110, 2014.

– S.-R. Hu, S. Peeta and H.-T. Liou, "Integrated Determination of Network Origin–Destination Trip Matrix and Heterogeneous Sensor Selection and Location Strategy," in IEEE Transactions on Intelligent Tranportation Systems, 2015.

– Fu, N. Zhu, S. Ling, S. Ma and Y. Huang, "Heterogeneous sensor location model for path reconstruction," Transportation Research Part B, p. 77–97, 2016.

– M. Gentili and P. B. Mirchandani, "Review of optimal sensor location models for travel time estimation," Transportation Research Part C 90, p. 74–96, 2018.

– N. Zhu, F. Fu and S. Ma, "Data-driven distributionally robust optimization approach for reliable travel-time-information-gain-oriented traffic sensor location model," Transportation Research Part B 113, p. 91–120, 2018.

– H. Yang, Y. Iida and T. Sasaki, "An analysis of the reliability of an origin-destination trip matrix estimated from traffic counts," Transportation Research Part B 25, pp. 351-363, 1991.

– H. Yang, C. Yang and L. Gan, "Models and algorithms for the screen line-based traffic-counting location problems," Computers & Operations Research 33, p. 836–858, 2006.

– L. U. Pando, R. Luders, K. V. O. Fonseca and M. O. Rosa, "Influence of Virtual Road Traffic Sensors of Oporto for Origin-Destination Matrix Estimation," in LADaS 2018 - Latin America Data Science Workshop, Rio de Janeiro, Brazil, 2018.

– IGENT TRANSPORTATION SYSTEMS, 2016.

– Castillo, A. Cobo, F. Jubete, R. E. Pruneda and C. Castillo, "An Orthogonally Based Pivoting Transformation of Matrice and Some Applications," Siam Journal of Matrix analysis and Applications, pp. 666-681, 2000.

– O. Christensen, An Introduction to Frame and Riesz Bases, Boston: Birkhäuser - Springer Science + Business Media, 2003.

– J. Kovacevic and A. Chebira, "An Introduction to Frames," Foundations and Trends in Signal Processing, p. 1–94, 2008.

– P. G. Casazza and G. Kutyniok, Finite Frames, Theory and Applications, New York: Springer Science+Business Media, 2013.

– M. Hadavi and Y. Shafahi, "Vehicle identification sensor models for origin–destination estimation," Transportation Research Part B - 89, p. 82–106, 2016.

– M. Gentili and P. Mirchandani, "Locating Active Sensors on Traffic Networks," Annals of Operations Research volume 136, p. 229–257, 2005.

– Castillo, P. Jimenez, J. Menendez and J. Conejo, "The observability problem in traffic models: algebraic and topological methods," IEEE Transactions on Intelligent Transportation Systems, Vol 9, no. 2, p. 275–287, 2008.

– Z. Sun and X. (. Ban, "Vehicle trajectory reconstruction for signalized intersections using mobile traffic sensors," Transportation Research Part C 36, p. 268–283, 2013.

– M. P. Dixon, "Real-Time OD Estimation Using Automatic Vehicle Identification and Traffic Count Data," Computer-Aided Civil and Infrastructure Engineering 17, p. 7–21, 2002.

– H. Park and A. Haghani, "Optimal number and location of Bluetooth sensors considering stochastic travel time prediction," Transportation Research Part C 55, pp. 203-216, 2015.

– S. Bekhor, M. E. Ben-Akiva and M. S. Ramming, "Evaluation of choice set generation algorithms for route choice models," Annals of Operations Research 144, p. 235–247, 2006.

– R. B. Dial, "A Probabilistic Multipath Assignment Model which Obviates Path Enumeration." Transportation Research 8, pp. 83-111, 1971.

– M. Ben-Akiva, M. J. Bergman, A. J. Daly and R. Ramaswamy, "Modelling inter urban route choice behaviour," in Ninth International Symposium on Transportation and Traffic Theory, Delft - The Netherlands, 1984.

– T. de la Barra, B. Pérez and J. Anez, "Multidimensional path search and assignment," in 21st PTRC Summer Meeting, 1993.

– J. Azevedo, M. Costa, J. Madeira and E. Martins, "An algorithm for the ranking of shortest paths," European Journal of Operational Research, 69 (1), pp. 97-106, 1993.

– M. Ramming, Network Knowledge and Route Choice, PhD thesis, Massachusetts Institute of Technology, 2001.