Incremental Layerwise Finite Element Formulation for Viscoelastic Response of Multilayered Pavements

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Abstract

This paper provides an incremental layerwise finite element formulation for the viscoelastic analysis of multilayered pavements. The constitutive behavior of asphalt concrete is represented by the Prony series. Layerwise finite element has been shown to provide an efficient and accurate tool for the numerical simulation of laminated structures. Most of the previous researches on numerical simulation of laminated structures have been limited to elastic material behavior. Thus, the current work focuses on layerwise finite element analysis of laminated structures with embedded viscoelastic material such as pavements.A computer code based on the presented formulation has been developed to provide the numerical results. The proposed approach is verified by comparing the results to the analytical solutions, existing numerical solutions in the literature, and those obtained from the ABAQUS software, as well.Finally, and as an application of the presented formulation, the effects of time and load rate on the quasi-static structural response of asphalt concrete (AC) pavements are studied.

Keywords: Layerwise finite element; linear viscoelastic materials; Multilayered structures; Pavements; Prony series

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1. Introduction

Evaluating pavement responses, such as stresses and strains, in pavement layers, is an important issue for pavement designers. Since the pavement is a multilayered system, the analysis of flexible pavements is a complex problem. Conventionally, the pavement is considered as a layered elastic structure for which the stresses and strains can be estimated based on the layered elastic theory [Yoder and Witczak,1975]. The initial studies of structural analysis in pavements have been carried out by Boussinesq [Boussinesq,1885] in which soils were modeled as a linear-elastic material. Afterwards the Boussinesq's theory was extended to a multilayer elastic model and Burmister [Burmister,1943, 1945] was the first researcher who introduced solutions for two- and three-layer systems. Layered elastic theory has been systemized and widely used for analysis and design of flexible pavements. In the last decades, the theory has been extended into an arbitrary number of layers, and various computer codes have been developed, codes such as BISAR [Jong, Peutz and Korswagen,1979], KENLAYER [Huang,1993], JULEA [Uzen,1994] and BISAR [Hildebrand, 2002]. However, most of the previous research works have been limited to elastic material behavior.; It is well known that asphalt concrete is a complex multilayered structure where the surface layer has a time-dependent behavior due to the viscoelastic binder. Therefore, the elasticity assumption is valid only when the asphalt mixture stays below the glass transition temperature,[Jaeseung, Reynaldo and Thomas,2009]. Obtaining more realistic structural responses for asphalt mixture is still a challenging topic in mechanistic pavement design methods. Accurate simulation of these types of materials requires the use of viscoelastic constitutive models[Jianlin and Bjorn, 2007]. The main problem towards mathematical modeling of the mechanical behavior of these materials is time dependency, i.e. the response is not only a function of the current input but also depends on the past input history and as a consequence, the solution of such problems becomes more complicated than the relevant linear elastic problem. For this reason, sometimes the viscoelastic solutions for simple problems are estimated by using the associated elastic solutions. For this purpose, the viscoelastic equations are transformed into equivalent elastic ones by means of Fourier and Laplace transforms. After solving the transformed problem, a numerical inversion is employed to recover the desired time domain response (see for example [Bozzaand Gentili,1995; Drozdov and Dorfmann,2004;Chou andLarew,1968; Huang,1973; Lee,1955;Hopman,1996;Radok,1957]).

Due to the complexity of the viscoelastic constitutive relations; however, such treatment is reasonable only in the cases of simple geometries or idealized boundary conditions. In addition, in the cases for which the relaxation or creep molduli cannot be expressed accurately by a simple model, the viscous parameters are time dependent, or complicated time dependent boundary conditions should be implemented, the Laplace transformation-inversion method become too complicated to be used. Since most of the existing analytical solutions require retaining of the complete history of stress and strain in the memory of a digital computer, these methods fail to deal with real and complex problems [Ghazlan, Caperaa and Petit,1995; Zocher, Groves and Aellen,1997]. To overcome this shortcoming, a number of theories based on the incremental constitutive equations have been proposed. For example, incremental constitutive equations in conjunction with finite element method have been employed by some researchers [Ghazlan, Caperaa and Petit,1995; Zocher, Groves and Aelle,1997; Kim and Sung Lee, 2007; Theocaris ,1964; Chazal, Pitti, 2009; Dubois; Chazal and Petit, 1999].

Different types of mathematical models are usually obtained for the viscoelastic behavior of materials by means of different arrangements of spring(s) and dashpot(s). For example, Generalized models (Prony series models) such as the generalized Maxwell model consisting of a spring and Maxwell elements connected in parallel manner [Zocher, Groves and Aelle,1997; Elseifi, Al-Qadi and Yoo, 2006; Chen, Pan and Green, 2009; Gibson et. al., 2003], the generalized Kelvin model consisting of a spring and N Voigt elements connected in series [Ghazlan, Caperaa and Petit,1995; Chazal, Pitti,2009; Dubois, Chazal and Petit, 1999; Park and Shapery, 1999], sigmoidal models [Pellinen and Witczak, 2002], parabolic models [Olard et. al., 2005; Benedetto et. al., 2004;Chupin et. al., 2010], power law model [Bahia and Anderson,1995], etc.…. In the present paper, a Prony- generalized Maxwell model series is adopted, which has been frequently employed by the researchers for modeling the asphalt concrete behavior [Elseifi, Al-Qadi and Yoo, 2006; Chen, Pan and Green, 2009; Gibson et. al., 2003; Park and Shapery, 1999; Junior et. al., 2010; Fujie and Walubita, 2006].

A displacement based layerwise finite element method (LW-FEM) has been introduced by Reddy [Reddy,1987] to accurately determine the transverse shear and normal stress distributions of laminated composites based on the three-dimensional elasticity theory. This method was increasingly used by others as in the original form proposed by Reddy or in its extended forms for the solution of engineering problems, such as laminated structure [Lee and Liu, 1992; Malekzadeh, 2009; Setoodeh and Karami, 2004; Malekzadeh, Setoodeh and Barmshouri, 2008; Setoodeh, Malekzadeh and Nikbin, 2009] during recent years. The layerwise theory is a refined theory that can take into account the thickness effects with minimum computational cost [Malekzadeh, 2009; Setoodeh and Karami, 2004; Malekzadeh, Setoodeh and Barmshouri, 2008; Setoodeh, Malekzadeh and Nikbin, 2009]. Since the layerwise theories assume a separate displacement field expansions within each subdivision, it provides a kinematically correct representation of the strain field in discrete layers [Reddy,1987; Bert and Malik,1996]. In addition, because the shear strains are discontinuous, this leaves the possibility of the continuous transverse stresses between adjacent layers in the layerwise theory. The approach actually shrinks the modeling to a combined from two one-dimensional analysis, which considerably reduces the number of manipulations, and the complexity in the formulation of the 2D finite element method. Implementing such methodology is not possible in powerful software like ABAQUS, which are developed on the basis of the conventional FEM.

The current work presents a simple and efficient algorithm based on the incremental viscoelastic LW-FEM for axisymmetric analysis of multilayer viscoelastic structures and in particular, asphaltpavements. This paper presents the details of the LW-FE formulation, verification and an asphalt pavement simulation example. The model is capable of evaluating the contact force, displacement, velocity and stress time histories for laminated structures subjected to different boundary conditions. Apart from simulation of asphalt pavements, the present approach could also be used for analysis of other engineering problems that exhibit viscoelastic behavior.

2. Characterization of Viscoelastic Behavior of Asphalt Concrete

Asphalt concrete belongs to the wide group of viscoelastic materials. Usual laboratory tests are used to determine the linear viscoelastic characteristic parameters of asphalt mixtures such as relaxation modulus *C(t)*, creep compliance *D(t)*, and complex modulus *E**. The relaxation modulus, *C(t)*, is the ratio of the stress response to a constant strain input, whereas the creep compliance, *D(t)*, is the ratio of the strain response to a constant stress input. For purely elastic asphalt mixtures, *C(t)* and *D(t)* are reciprocals. However, as a result of viscoelastic characteristics of asphalt mixtures, this is only true in the Laplace transform domain [Chen, Pan and Green, 2009]. The viscoelastic behavior of asphalt concrete can be modeled by either the generalized Maxwell model, or the generalized Kelvin model [Elseifi, Al-Qadi and Yoo, 2006;Gibson et. al., 2003].The relaxation modulus *C(t)* from the generalized Maxwell model and the creep compliance *D(t)* from the generalized Kelvin model are given by [Chen, Pan and Green, 2009].

$$C_{ijkl}(t)= C^{(\infty)}_{ijkl}+\sum_{m=1}^{M} C^{(m)}_{ijkl}\, e^{\overline{\frac{t}{\rho^{(m)}_{ijkl}}}}\; H(t),\; D_{ijkl}(t)= D^{(0)}_{ijkl}+\sum_{n=1}^{N} D^{(n)}_{ijkl}\, 1e^{\overline{\frac{t}{\tau^{(n)}_{ijkl}}}}\, H(t) \qquad (1\text{ a, b})$$

where:

$C^{(\infty)}_{ijkl}$ and $C^{(m)}_{ijkl}$ are equilibrium modulus and relaxation strength, respectively;

$D^{(0)}_{ijkl}$ and $D^{(n)}_{ijkl}$ are glassy compliance and the retardation strength receptively; $\rho^{(m)}_{ijkl}$ and $\tau^{(n)}_{ijkl}$ are relaxation time and retardation time, respectively;

H(t) is the unit step function.

Once viscoelastic materials represent time dependency, their responses at a given instant do not only depend on the applied load at that instant but also on the complete history [Christensen, 1982]

Using the Boltzmann superposition principle [Boltzmann,1878], the stress-strain relationship of a viscoelastic material can be written as a hereditary integral. For instance, the stress $\sigma_{ij}(t)$ and strain $\varepsilon_{ij}(t)$ of a linear viscoelastic material for 2D problems is given by:

$$\sigma_{ij}(t)=\sum_{k}^{2}\sum_{l}^{2}\int^{t} C_{ijkl}(t\quad\tau)\frac{\partial\,\varepsilon_{kl}}{\partial\quad\tau}\,d\tau\ ,\quad \varepsilon_{ij}(t)=$$

$$\sum_{k}^{2}\sum_{l}^{2}\int^{t} D_{ijkl}(t\quad\tau)\frac{\partial\,\sigma_{kl}}{\partial\quad\tau}\,d\tau \qquad (2\text{ a, b})$$

where $C_{ijkl}(t)$ and $D_{ijkl}(t)$ are the components of relaxation modulus and creep compliance tensor, respectively,τ is the time dummy variable and t is the time since loading. Hereafter, repeated indices imply the summation convention.

From Eq. (2 a, b) it is obvious that the response is a function of the current input and the input history as well. The constitutive equations represented by (2 a, b) can be expressed into an incremental form in order to be used with a layerwise finite element analysis. This method is based on a time discretization of the compliance or relaxation function according to Prony series represented by (1 a, b). The use of incremental form has the advantage of eliminating the memory possessing problem because it does not need to solve a set of differential equations simultaneously. Different time integration algorithms for the linear viscoelastic problems have been presented in the literature for both isotropic and anisotropic solids [Ghazlan, Caperaa and Petit, 1995; Zocher, Groves and Aellen, 1997; Taylor, Pister and Gourdreau, 1970]. In this study, after a comprehensive review of the relevant literature, the numerical algorithm proposed by Zocher [Zocher, Groves and Aellen,1997] was adopted. Based on this method, the relaxation function $C(t)$ is used to display the Prony series, while in the Ghazlan's method [Ghazlan, Caperaa and Petit,1995], the creep compliance $D(t)$ is used. Based on this algorithm and prior to development of the layerwise finite element formulation, the constitutive relation in terms of relaxation functionis transformed from integral form into an incremental algebraic form by a Prony series representation. In the following, the main formulations and principles of this algorithm are briefly described. In this algorithm, the time corresponding to the next step n+1 is obtained from the current time t_n through:

$$\Delta t_n = t_{n+1}\quad t_n \qquad (3)$$

where:

Δt_n is a time increment between the two steps.

The stress increment from time t_n to time t_{n+1} is defined as,

$$\Delta\sigma_{ij}=\Delta\sigma^{R}_{ij}+\sum_{k}^{2}\sum_{l}^{2}\overline{C}_{ijkl}(\Delta\varepsilon_{kl}) \qquad (4)$$

where:

$\Delta\varepsilon_{ij}$ and $\Delta\sigma_{ij}$ are the increments of strain and stress tensor components, respectively, and $\overline{C}_{ijkl}$ is a fourth order tensor which can be interpreted as a viscoelastic relaxation tensor,

$$\overline{C}_{ijkl}=C^{(\infty)}_{ijkl}+\frac{1}{\Delta t_n}\sum_{m}^{M}\rho^{(m)}_{ijkl}C^{(m)}_{ijkl}\ \ 1\ \ e^{\frac{\Delta t_n}{\rho^{(m)}}} \qquad (5)$$

Also,$\Delta\sigma^{R}_{ij}$ is given by

$$\Delta\sigma^{R}_{ij}=\sum_{m}^{M}1\ e^{\frac{\Delta t_n}{\rho^{(m)}_{ijkl}}}\ S^{(m)}_{ijkl}(t_n) \qquad (6)$$

where $S^{(m)}_{ijkl}$ can be expressed as

$$S^{(m)}_{ijkl}(t_n)=e^{-\frac{\Delta t_n}{\rho^{(m)}_{ijkl}}}S^{m}_{ijkl}(t_{n-1})+\rho^{(m)}_{ijkl}C^{(m)}_{ijkl}\,\dot{\varepsilon}_{kl}(t_{n-1})\left(1-e^{-\frac{\Delta t_n}{\rho^{(m)}_{ijkl}}}\right) \qquad (7)$$

where $\dot{\varepsilon}_{kl}(t_{n\ 1})$ is constant representing the time rate of change over the interval from time step t_{n-1} to t_n . When using small time increments, it is possible to assume a constant strain rate variation during the interval

$$\varepsilon_{kl}=\frac{\partial\ \varepsilon_{kl}}{\partial t}\quad\frac{\Delta\varepsilon_{kl}}{\Delta t} \qquad (8)$$

For a complete solution of the algorithm regarding the incrementalization method, refer to [Zocher, Groves and Aellen, 1997]. The incremental constitutive law represented by (4) can be introduced in a layerwise finite element discretization in order to obtain solutions to complex viscoelastic problems.

3. Axisymmetric LayerwiseFinite Element Formulation for Viscoelastic Analysis

3.1 Axisymmetric Modeling

An axisymmetric analysis is carried out for the LW-FE approach. The axisymmetric modeling has been selected because it could simulate circular loading and did not require excessive computational cost. A typical axisymmetric cross section is shown in Figure1. Based on the two-dimensional theory of elasticity, the linearized axisymmetric strain displacement relations are as follows,

$$\{\varepsilon_{ij}\}^T=\{\varepsilon_{rr}\ \ \varepsilon_{zz}\ \ \varepsilon_{\theta\theta}\ \ \Upsilon_{rz}\}=\ \frac{\partial u}{\partial r}\ \frac{\partial w}{\partial z}\ \frac{u}{r}\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r} \qquad (9)$$

where ε_{ii} (i=r , z,θ) and Υ_{xz} are the Lagrangian normal and shear components of the strain tensor, respectively.

3.2 Layerwise Finite Element Formulation for Viscoelastic Analysis

Based on this theory, the laminated structure in the thickness direction is subdivided into a series of N_{ML} mathematical layers. In each mathematical layer, the displacement components in the thickness direction are approximated in a similar manner as the one-dimensional finite element method. Total number of nodes in the z-direction, N_z ,is determined in terms of the number of mathematical layers N_{ML} and the node per layer N_{PL} according to $N_z = (N_{PL} - 1)N_{ML}+1$. The layerwise concept is general such that the number of subdivisions through the thickness can be greater than, equal to, or less than the number of material or physical layers through the thickness. Independent shape functions are defined through the layer thickness as well as the longitudinal direction. Any desired displacement variation degree through the thickness is easily obtained by either adding more 1-D finite element subdivisions along the thickness, or using higher–order Lagrangian interpolation polynomials through the thickness. In this study this study, the linear global shape functions $\psi_j(z)$ and $_i(r)$ are used. For the case of two nodes per mathematical layers in the thickness direction and r-direction elements, the global interpolation functions become:

$$\psi_j(z)=\begin{cases} 0 & Z\le Z_{j-1} \\ \dfrac{z-z_{j-1}}{z_j-z_{j-1}} & Z_{j-1}\le Z\le Z_j \\ \dfrac{z_{j+1}-z}{z_{j+1}-z_j} & Z_j\le Z\le Z_{j+1} \\ 0 & Z_{j+1}\le Z \end{cases} \quad \text{for } j=1,2,3,...,N_z \qquad (10)$$

And

$$\varphi_i(r)=\begin{cases} 0 & r\le r_{i-1} \\ \dfrac{r-r_{i-1}}{r_i-r_{i-1}} & r_{i-1}\le r\le r_i \\ \dfrac{r_{i+1}-r}{r_{i+1}-r_i} & r_i\le r\le r_{i+1} \\ 0 & r_{i+1}\le r \end{cases} \quad \text{for } i=1,2,3,...,N_r \qquad (11)$$

where r_i and z_j are the *r*-coordinate and *z*-coordinate of the global node *i* and *j*, respectively. $_i(r)$ and $_j(z)$ are the 1D the global interpolation function in the r and z-direction, respectively.

In an incremental displacement-based formulation, the deformation would be stated in terms of these shape functions and the nodal displacements. Hence, the incremental displacement components of any desired element at any point within the j^{th}-layer may be expressed as,

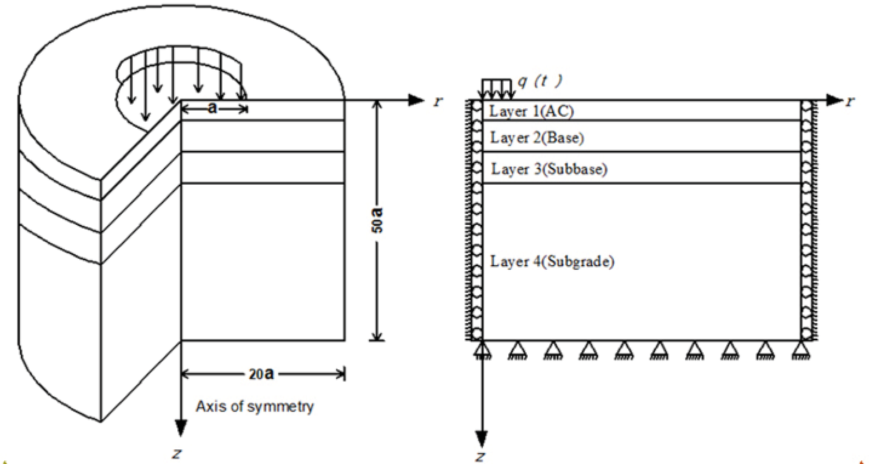


Figure 1. Pavement section, loading and boundary conditions.

$$\Delta u_r = \Delta u(r,z,\Delta t_n) = \sum_{i=1}^{N_r}\sum_{j=1}^{N_z} \Delta U_{ij}(\Delta t_n)\, \varphi_i(r)\psi_j(z) = \Delta U_{ij}(\Delta t_n)\, \varphi_i(r)\psi_j(z) \quad (12)$$

$$\Delta u_z = \Delta W(r,z,\Delta t_n) = \sum_{i=1}^{N_r}\sum_{j=1}^{N_z} \Delta W_{ij}(\Delta t_n)\, \varphi_i(r)\psi_j(z) = \Delta W_{ij}(\Delta t_n)\, \varphi_i(r)\psi_j(z) \quad (13)$$

where Δu_r and Δu_z are the incremental displacement components of an arbitrary material point in the r–z plane along r and z directions, respectively; $\Delta U_{ij}(\Delta t_n)$ and $\Delta W_{ij}(\Delta t_n)$ are the incremental displacement components of i^{th} node corresponding to j^{th} node (defined by $r=r_i$ and $z=z_i$) in the r and z-direction, respectively; also, as obvious from Eqs. (12) and (13), for brevity purpose the indicial summation notation is used.

To derive the incremental stress-strain relations at an arbitrary point of a laminate, the axisymmrtric constitutive relations is used, which according to Eq. (4) for a viscoelastic materials is

$$\begin{Bmatrix} \Delta\sigma_{rr} \\ \Delta\sigma_{zz} \\ \Delta\sigma_{\theta\theta} \\ \Delta\sigma_{rz} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 \\ 0 & 0 & 0 & \bar{C}_{44} \end{bmatrix}_{\Delta t_n} \begin{Bmatrix} \Delta\varepsilon_{rr} \\ \Delta\varepsilon_{zz} \\ \Delta\varepsilon_{\theta\theta} \\ \Delta\gamma_{rz} \end{Bmatrix} + \begin{Bmatrix} \Delta\sigma^R_{rr} \\ \Delta\sigma^R_{zz} \\ \Delta\sigma^R_{\theta\theta} \\ \Delta\sigma^R_{rz} \end{Bmatrix}$$

$$= [\bar{C}]_{\Delta t_n}\{\Delta\varepsilon_{ij}\} + \{\Delta\sigma^R_{ij}\} \quad (14)$$

where: $\Delta\sigma_{ii}$ ($i=r, z, \theta$) and $\Delta\sigma_{xz}$ are the Lagrangian increments of normal and shear stress, respectively. Also $[\bar{C}]_{\Delta t_n}$ is the viscoelastic material stiffness matrix in the principal coordinates of the laminae. Substituting the displacement components from Eqs. (12) and (13) into Eq. (9), -results:

$$\Delta\varepsilon_{rr}(\Delta t_n) = \Delta U_{ij}(\Delta t_n)\psi_j(z)\frac{d\varphi_i(r)}{dr}$$

$$\Delta\varepsilon_{zz}(\Delta t_n) = \Delta W_{ij}(\Delta t_n)\varphi_i(r)\frac{d\psi_j(z)}{dz} \quad (15\text{ a- d})$$

$$\Delta\varepsilon_{\theta\theta}(\Delta t_n) = \frac{\Delta U_{ij}(\Delta t_n)}{r}$$

$$\Delta\gamma_{rz} = \Delta U_{ij}(\Delta t_n)\varphi_i(r)\frac{d\psi_j(z)}{dz} + \Delta W_{ij}(\Delta t_n)\psi_j(z)\frac{d\varphi_i(r)}{dr}$$

The incremental equilibrium equations for the viscoelastic problem under consideration can be derived using the principle of virtual work. For a two-dimensional continuum problem, during the time step Δt_n in dynamic condition can be written as follows:

$$2\pi\int_0^R\int_0^h [\Delta\sigma_{rr}\delta(\Delta\varepsilon_{rr}) + \Delta\sigma_{zz}\delta(\Delta\varepsilon_{zz}) + \Delta\sigma_{\theta\theta}\delta(\Delta\varepsilon_{\theta\theta}) + \Delta\sigma_{rz}\delta(\Delta\gamma_{rz})]r\, dz\, dr$$

$$-2\pi\int_0^R \Delta q_T(r,t)\varphi(r)\delta_{jN_z}\delta(\Delta W_{ij})r\, dr - 2\pi\int_0^R \Delta q_R(z,t)\psi_j(z)\delta_{iN_r}\delta(\Delta U_{ij})r\, dr \quad (16)$$

$$+2\pi\int_0^R\int_0^h \rho(\Delta\ddot{U}\,\delta(\Delta U_{ij}) + \Delta W_{ij}))r\, dz\, dr = 0$$

where: dot over a displacement components represent its derivative with respect to time; $\Delta q_T(r,t)$ and $\Delta q_R(z,t)$ are the increments of external transverse and radial loads, ρ the material density, R the domain radius, h the domain depth; and finally δ_{ij} is the Kronecker delta. Using Eqs. (14)–(16) and performing the finite element assembling procedure, the discretized equations of motion take the following form,

$$\begin{bmatrix} [K_{uu}] & [K_{uw}] \\ [K_{wu}] & [K_{ww}] \end{bmatrix}_{\Delta t_n} \begin{Bmatrix} \{\Delta U\} \\ \{\Delta W\} \end{Bmatrix} + \begin{bmatrix} [M_{uu}] & [0] \\ [0] & [M_{ww}] \end{bmatrix}_{\Delta t_n} \begin{Bmatrix} \{\Delta\ddot{U}\} \\ \{\Delta\ddot{W}\} \end{Bmatrix} = \begin{Bmatrix} \Delta f_u \\ \Delta f_w \end{Bmatrix} - \begin{Bmatrix} f^R_u \\ f^R_w \end{Bmatrix}_{t_n} \quad (17)$$

where: $\begin{matrix}\Delta f_u \\ \Delta f_w\end{matrix}$ is the external load vector increment and $\begin{matrix}f^R_u \\ f^R_w\end{matrix}$ is the elements load vector due to change of stresses during the time. The elements of the stiffness matrices $[K_{ij}]$(i ,j=u, v), the mass matrices $[M_{ij}]$ (i ,j=u, v), and the load vector increments $\begin{matrix}\Delta f_u \\ \Delta f_w\end{matrix}$ and $\begin{matrix}f^R_u \\ f^R_w\end{matrix}$ are presented in Appendix A.

In problems for which the loading is applied quasi-statically (i.e. the dynamic effects of the acceleration are negligible), the pertinent terms can be dropped from the equation of motionto reduce Eq. (17)to

$$\begin{bmatrix} [K_{uu}] & [K_{uw}] \\ [K_{wu}] & [K_{ww}] \end{bmatrix}_{\Delta t_n} \begin{Bmatrix} \{\Delta U\} \\ \{\Delta W\} \end{Bmatrix} = \begin{Bmatrix} \Delta f_u \\ \Delta f_w \end{Bmatrix} - \begin{Bmatrix} f^R_u \\ f^R_w \end{Bmatrix}_{t_n} \quad (18)$$

4. Code Development

Based on the theoretical concept discussed in the preceding sections, a code was developed in this study. The base part of this code is anaxisymmetric LW-FE program as defined previously. An outline of the step-by-step calculations performed in the code is provided as follows. The solution algorithm is also shown in Figure 2.

Suppose that mechanical fields are known at time t_n and the time increment is fixed to Δt_n .

1. Firstly, the relaxation tensor $\overline{C_{ijkl}}$, is computed from Eq. (5) and the global linear viscoelastic stiffness matrix $[K_{ij}]_{\Delta t_n}$ is evaluated from Eqs. (A1-A12)

2. Pseudo stress tensor is computed from Eq. (6) and then the load vector, is determined using Eq. (A15-A17).

3. The equilibrium equations (18) are solved for the external load vector $\begin{Bmatrix} \Delta f_u \\ \Delta f_w \end{Bmatrix}$ and the nodal displacement vector increment $\begin{Bmatrix} \{\Delta U\} \\ \{\Delta W\} \end{Bmatrix}$ isdetermined.

4. The strain increment $\Delta\varepsilon_{ij}$ is computed from the equations (15a-d).

5. The results of step (2) are utilized to compute the stress increment $\Delta\sigma_{ij}(\Delta t_n)$ using Eq. (4).
6. The state of all filed variables (displacement, stress and strain) is updated at the end of the time increment Δt_n as follows:

$$\{u(t_{n+1})\} = \{u(t_n)\} + \{\Delta u(\Delta t_n)\}$$

$$\{\varepsilon_{ij}(t_{n+1})\} = \{\varepsilon_{ij}(t_n)\} + \{\Delta\varepsilon_{ij}(\Delta t_n)\}$$

$$\{\sigma_{ij}(t_{n+1})\} = \{\sigma_{ij}(t_n)\} + \{\Delta\sigma_{ij}(\Delta t_n)$$

8. If time Step n+1 is the last step, do nothing. Otherwise, $\Delta\sigma_{ij}^{R}(t_{n+1})$ is calculated using the resulted $\sigma_{ij}(t_{n+1})$, $\varepsilon_{ij}(t_{n+1})$ and Eqs.(6), (7).

9. Go to step 1

5. Numerical Results

In this section, the presented formulation and method of solution is validated and then a pavement structure composed of a viscoelastic layer and three elastic layers is analyzed and some numerical results are presented. In order to evaluate the accuracy of the axisymmetric layerwise finite element formulation, the numerical solution is compared to analytical solutions for three-layer pavement. The stress values calculated using the proposed algorithms are compared to the widely used Jones' Tables of stresses in three-layer elastic system [Jones,1962]. Pavement structure and material properties for this set of analysis is given in Table 1, pavement section is subjected to a circular load with a radius of 15 cm and uniform pressure of 550KPa. The LW-FE domain has the size of 50×a in the verticaldirection and 20×a in the horizontal directions, where a is the radius of circular loaded area .

Table 2, shows the stresses calculated using LW-FE formulation and analytical solution by Jones. One can observe that the results are in excellent agreement with those of analytical solution.

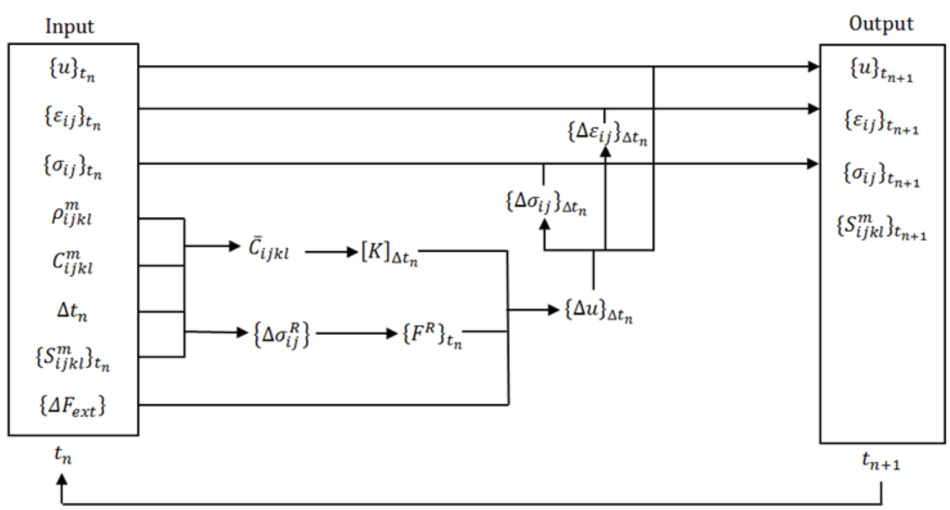


Figure 2. Solution algorithm

Table 1. Pavement structure and material properties for validation of axisymmetric LW-FE analysis

Pavement layer	E(MPa)	Poisson ratio	Thickness(m)
Layer1 ($N_{ML}=7$)*	4000	0.5	0.075
Layer2 ($N_{ML}=10$)	200	0.5	0.30
Layer3 ($N_{ML}=25$)	100	0.5	7.125

* Number of mathematical layers per physical layer.

Table 2. Predicted stressesby LW-FE and Jones(Number of radial element=32).

stress	LW-FE(MPa)	Jones(MPa)
σ_{zz1} (top of layer2)	0.26740	0.2550
σ_{rr1} (bottom of layer1)	-1.9730	-1.9875
σ_{zz2} (top of layer3)	0.05564	0.05492
σ_{rr2} (bottom of layer2)	-0.05745	-0.05884

After analytical validation of the proposed approach, a multilayered pavement section, with a viscoelastic layer, is analyzed and the results are compared to the ones obtained using the software ABAQUS. The pavement section (see Figure 1) is simulated by using an axisymmetric layerwise finite element model. The pavement structure is composed of four homogeneous and isotropic layers. In order to incorporate the viscoelastic properties of the asphalt concrete, the top layer is assumed to be viscoelastic, while the lower three layers have linear elastic behavior. The viscoelastic properties of the top layer are characterized by the time-dependent relaxation modulus *C(t)* and Poisson's ratio. It is assumed that the layers are perfectly bonded and thus, the tractions and displacements are continuous across the interface. The time-dependent load $\hat{q}(t)$ is applied over a circle of radius a on the surface z=0. In this study, a step load is applied on the surface, as follows:

$$\sigma_{zz}(r,z=0)=\begin{cases}-\hat{q}(t)H(t) & r<a\\ 0 & r>a\end{cases} \tag{19}$$

The intensity of the load $\hat{q}(t)$ varies with the time of the half sine wave, as assumed by [Huang,1993],

$$\hat{q}(t)=q_0\sin^2\left(\frac{\pi t}{t_0}\right) \tag{20}$$

where q_0 is the load amplitude and t_0 is the load duration.

Pavement structure is subjected to a circular load which has radius of 15cm and pressure of 550KPa.LW-FE domain has the size of 50×a(7.5m) in the vertical direction and 20×a(3m) in the horizontal directions, where a is the radius of circular loading.

The pavement under study is composed of AC, base, granular sub-base and subgrade with parameters listed in Table 3, The AC layer is viscoelastic and material constants used in the relaxation functionpresented in Eq. (1) are listed in Table 4. The bottom surface of the subgrade is assumed to be fixed, which means that nodes at the bottom of the subgrade cannot move horizontally or vertically. The boundary nodes along the pavement edges are horizontally constrained, but are free to move in the vertical direction. Boundary conditions for the problem are given in Figure1.

A load duration $t_0=0.1$ s corresponding to the vehicle speed of 10 (km/h), is considered. Quasi-static analyses are conducted and computed parameters are vertical displacement at the top of the surface layer, radial and shear stress at the bottom of AC layer and vertical stress on the top of the subgrade layer. According to Table 5,the predicted pavement responses are in agreement with ABAQUS results.

Table 3. The material properties and thickness of layers for a typical four-layer pavement

Layer	Thickness (m)	Young's modulus (MPa)	Poisson's ratio
AC	0.15	Viscoelastic(table4)	0.35
Base	0.2	500	0.3
Subbase	0.2	200	0.3
Subgrade	6.95	100	0.3

Table 4. The values of C^m and ρ^m for viscoelastic AC layer[Lee,1996]

m	$C^m(\text{kPa})$	$\rho^m(s)$
∞	1172	–
1	3.10×10^{6}	2.20×10^{-5}
2	4.31×10^{6}	2.20×10^{-4}
3	3.46×10^{6}	2.20×10^{-3}
4	2.02×10^{6}	2.20×10^{-2}
5	1.27×10^{6}	2.20×10^{-1}
6	2.72×10^{5}	2.20×10^{0}
7	6.59×10^{4}	2.20×10^{1}
8	1.45×10^{4}	2.20×10^{2}
9	1.52×10^{3}	2.20×10^{3}
10	7.10×10^{2}	2.20×10^{4}
11	5.88×10^{1}	2.20×10^{5}

Table 5. Predicted Responses by LW-FE and ABAQUS[t=0.05 (s)].

Pavement responses	LW_FE[a]	ABAQUS[b]
W(0,0)(m)	-0.0003224	-0.00032378
$\sigma_{rr}(0,-0.15)$(Pa)	411864	404267
$\sigma_{rz}(0,-0.15)$(Pa)	12750	12767
$\sigma_{zz}(0,-0.55)$(Pa)	-24248	-24251

[a] $N_r \times N_z = 33\times43$ and $\Delta t = 0.001$

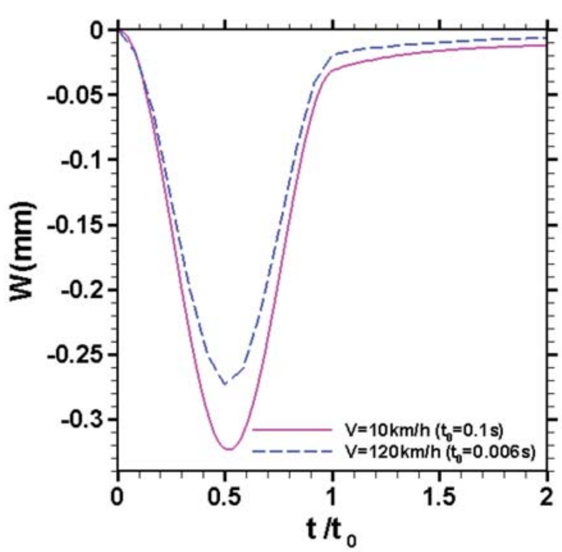
[b] 1970 linear quadrilateral elements of type CAX4R and $\Delta t = 0.001$

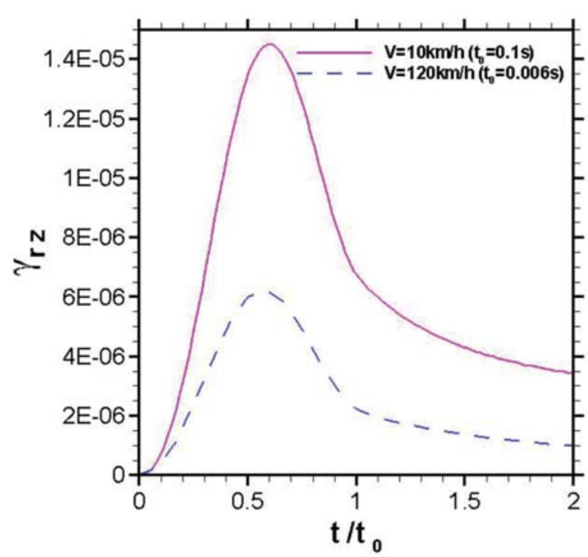
Using the presented formulation, to study the influence of different values of the load's duration *(t_0)* of 0.1 s and 0.006 s corresponding to speeds of 10 (km/h) and 120 (km/h) of the vehicle, respectively, the quasi-static analyses is conducted in order to evaluate the importance of velocity effects.

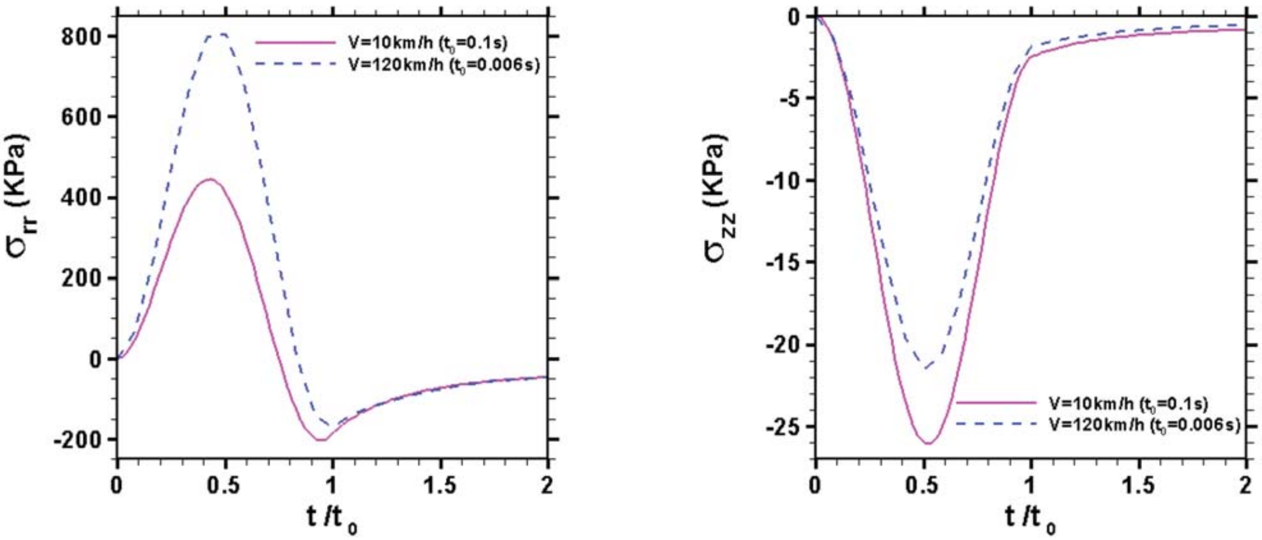
In Figures 3(a)-(d), the time histories of the symmetry axis displacement on the top surface *W(0,0)*, the symmetry axis shear stress $\gamma_{ra}(0,\ 0.15)$ and radial stress $\sigma_{rr}(0,\ 0.15)$ at the bottom of the AC layer, and the symmetry axis transverse normal stress applied on the top of subgrade $\sigma_{zz}(0,\ 0.55)$ for the two different load duration of 0.1 s (correspond to speed V =10 km/h) and 0.006 s (correspond to speed V=120 km/h) are presented. It is obvious that the load duration have significant effect on the variation of the symmetry axis shear stress $\gamma_{ra}(0,\ 0.15)$ and normal stress $\sigma_{rr}(0,\ 0.15)$ at the bottom of the AC layer. One of the most important characteristics of viscoelastic materials is the time lag between stress and strain. This is indicated in Figure 4 for $t_0=0.1$ s. Variation of the transverse normal strain $\varepsilon_{zz}(0,z)$ a and stress $\sigma_{zz}(0,z)$ along the symmetry axis of the pavement is shown in Figures 5 (a) and (b) for a pulse of 0.006s (V =120km/h). It can be seen that the presented formulation and method of solution can capture the zig-zag variation of the normal strain and the smooth variation of the transverse normal stress. Also, it is observable that the variation these quantities
have the same trend for both values of the load duration. Figures 6 (a) and (b) depict distribution through the depth for the radial strain $\varepsilon_{rr}(0,z)$ and stress $\sigma_{rr}(0,z)$ at the symmetry axis of the pavement for a pulse of 0.1s (V =10km/h) at t=0.05s.

6. Conclusions

An incremental layerwise finite element formulation was presented to solve the time-dependent response of multilayered viscoelastic pavement under surface loadings and a code was developed as well. The two-dimensional elasticity theory approach guarantees the generality of the method and the layerwise theory makes analysis economical and optimal. Incremental formulation avoids the numerical complexity in integral transform methods in the traditional numerical handling of viscoelasticity. The constitutive behavior of the viscoelastic layers was modeled using the Prony series in conjunction with the incremental formulation in the temporal domain. To validate the presented approach, the results were compared to the analytical and numerical solutions available in the literature. The quasi-static structural responses of the asphalt concrete pavements were studied and results were compared to ABAQUS software. As an important practical application, the quasi-static structural responses of the asphalt concrete pavements were studied and the effects of load duration on the deformations, strains and stresses were exhibited.







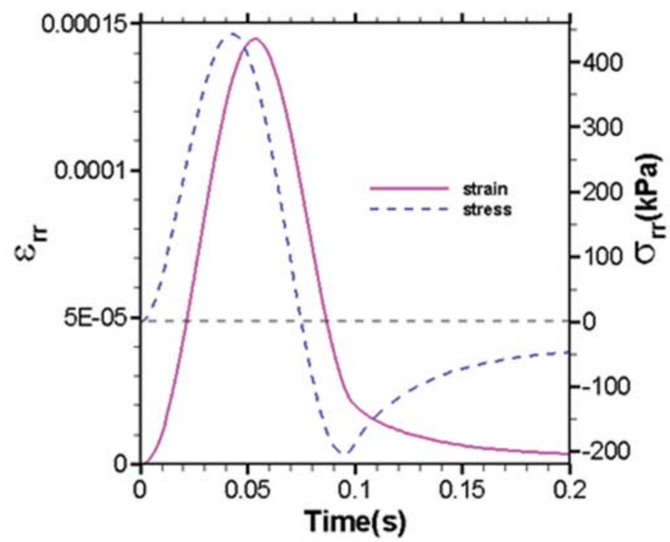


Figure 4. Stress and strain responses at the bottom of the surface layer , for a pulse of 0.1s (V =10km/h).

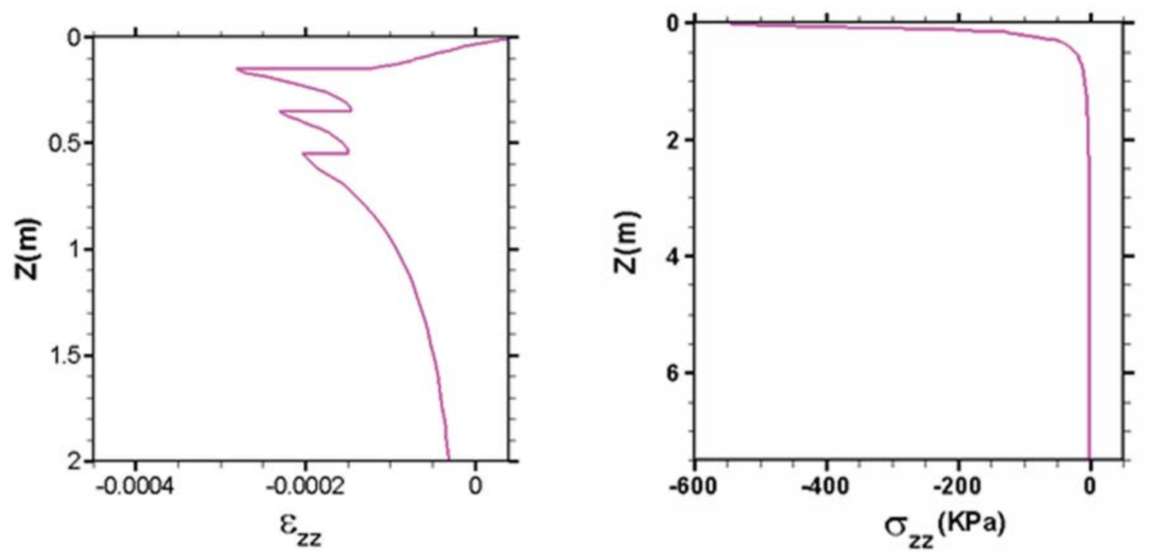


Figure 5. Distribution of the symmetry axis (r=0) transverse normal strain (ε_{zz}) and stress(σ_{zz}) in depth for a pulse of 0.006s (V =120km/h) at t=0.003s.

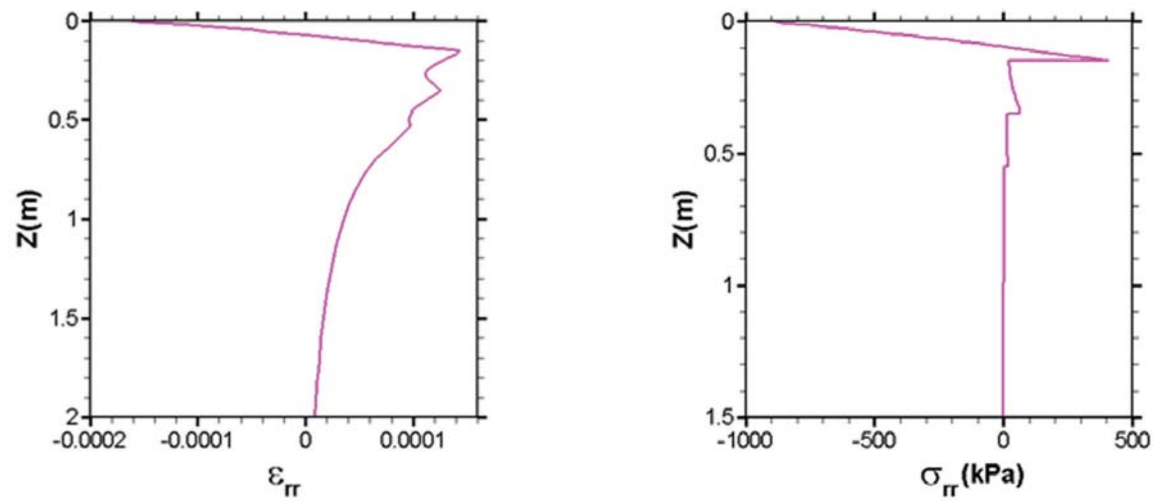


Figure 6. Distribution of the symmetry axis (r=0) radial strain (ε_{rr}) and stress (σ_{rr}) in depth for a pulse of 0.1s (V =10km/h) at t=0.05s.

7. References

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8. Appendix

The elements of the stiffness matrices $[K_{ij}]$, the mass matrices $[M_{ij}]$, the load vector increments $\begin{Bmatrix} \Delta f_u \\ \Delta f_w \end{Bmatrix}_{t_n}$ and $\begin{Bmatrix} f_u^R \\ f_w^R \end{Bmatrix}$ in Eq. (18) are

$$[K_{uu}] = A^{kr}_{p11} D^{ab}_{r} + A^{kr}_{p13} E^{ab}_{r} + B^{rk}_{p14} B^{ab}_{r} + A^{kr}_{p13} E^{ba}_{r} + A^{kr}_{p33} H^{ab}_{r} + B^{rk}_{p34} F^{ab}_{r} + B^{kr}_{p14} B^{ba}_{o} + B^{kr}_{p34} F^{ba}_{r} + D^{kr}_{p44} A^{ab}_{o},$$

$$[K_{uw}] = B^{rk}_{p12} B^{ab}_{o} + A^{kr}_{p14} D^{ab}_{c} + B^{rk}_{p23} F^{ab}_{o} + A^{kr}_{p34} E^{ab}_{c} + D^{rk}_{p24} A^{ab}_{o} + B^{kr}_{p44} B^{ba}_{o},$$

$$[K_{wu}] = B^{kr}_{p12} B^{ba}_{o} + B^{kr}_{p23} F^{ba}_{r} + D^{kr}_{p24} A^{ab}_{o} + A^{kr}_{p14} D^{ab}_{o} + A^{rk}_{p34} E^{ab}_{o} + B^{rk}_{p44} B^{ab}_{o},$$

$$[K_{ww}] = D^{kr}_{p22} A^{ab}_{r} + B^{kr}_{p24} B^{ba}_{r} + B^{rk}_{p24} B^{ab}_{c} + A^{kr}_{p44} D^{ab}_{o} \quad \text{(A.1a-d)}$$

$$[M_{uu}] = A^{ab}_{c} I^{kr}_{p},\ [M_{ww}] = A^{ab}_{o} I^{kr}_{p} \quad \text{(A.2a,b)}$$

$$A_{pqmn}^{ij} = 2\pi\int_0^h C_{pqmn}(\Delta t_n)\psi_j(z)\psi_r(z)dz \tag{A.3}$$

$$B_{pqmn}^{ij} = 2\pi\int_0^h C_{pqmn}(\Delta t_n)\psi_j(z)\frac{d\psi_r(z)}{dz}dz\,,\; B_{vqmn}^{ij} = 2\pi\int_0^h C_{vqmn}(\Delta t_n)\psi_r(z)\frac{d\psi_j(z)}{dz}dz \tag{A.4a,b}$$

$$D_{pqmn}^{jr} = 2\pi\int_0^h \overline{C}_{pqmn}(\Delta t_n)\frac{d\psi_j(z)}{dz}\frac{d\psi_r(z)}{dz}dz \tag{A.5}$$

$$A_c^{ls} = \int_0^R r\varphi_l(r)\varphi_s(r)dr \tag{A.6}$$

$$B_c^{ls} = \int_0^R r\frac{d\varphi_l(r)}{dr}\varphi_s(r)dr\,,\; B_c^{sl} = \int_0^R r\frac{d\varphi_s(r)}{dr}\varphi_l(r)dr \tag{A.7 a,b}$$

$$D_c^{ls} = \int_0^R r\frac{d\varphi_l(r)}{dr}\frac{d\varphi_s(r)}{dr}dr \tag{A.8}$$

$$E_c^{ls} = \int_0^R \frac{d\varphi_l(r)}{dr}\frac{\varphi_s(r)}{dr}dr \tag{A.9}$$

$$F_c^{ls} = \int_0^R \varphi_l(r)\varphi_s(r)dr \tag{A.109}$$

$$H_c^{ls} = \int_0^R \frac{1}{r}\varphi_l(r)\varphi_s(r)dr \tag{A.11}$$

$$I_p^{jr} = 2\pi\int_0^h \rho\,\psi_j(z)\psi_r(z)dz \tag{A.12}$$

$$\{\Delta f_u\}_{t_n} = \Delta f_j(t_n)\delta_{j\alpha r}\,;\; \{\Delta f_w\}_{t_n} = \Delta f_i(t_n)\delta_{j\alpha z} \tag{A.13a,b}$$