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Research Paper

Hub Location Problem in Public Transport, Considering Potential Hubs Establishment: A Bi-Objective Approach

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Abstract

The hub location problem in public transport focuses on strategically placing transport hubs to enhance service delivery and operational efficiency. Rising urban populations and transportation demands necessitate effective public transport systems. The challenge is to identify optimal locations for new hubs while balancing objectives like minimizing operational costs and minimize maximum passenger travel time. This study proposes a bi-objective model that selects potential hub locations from an interurban public transport network using the Multi-Criteria Decision-Making (MCDM) method of TOPSIS. The first objective is to minimize transportation costs, hub construction costs, and route creation costs, while the second aims to minimize maximum passenger travel time. Given that this problem is NP-hard, the Lagrangian Relaxation (LR) algorithm is employed for medium and large network sizes, with computational results provided. The proposed method is validated using Mandl's and Sioux Falls network data, which are standard benchmark datasets in transportation and network optimization. Results indicate that the algorithm effectively determines optimal hub locations. This approach can be adapted for specific parameters, bringing the problem closer to real-world conditions and uncertainties.

Keywords: Transportation, Bus network, Hub Location Problem, Lagrangian Relaxation Algorithm

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1. Introduction

Transportation systems are among the most fundamental sectors in any country. These systems play a critical role in the economic and social development of any region and can provide efficient ways to support future travels in urban areas. With the increasing population and the consequent increase in the demand for transportation, the importance of the proper design of transportation networks has become more apparent. According to the International Transport Forum [ITF, 2021[4]], urban public transport demand is projected to grow by 60% by 2050, with buses accounting for the dominant share in over 80% of developing cities. Moreover, the World Bank[5] (2020) reports that efficient public transportation systems can contribute up to 4% of national GDP through improved mobility, labor productivity, and reduced congestion. These trends underscore the strategic role of transit network design in addressing both economic and sustainability challenges in cities.

A widely adopted structural solution is the hub-and-spoke model, in which transfer activity is concentrated at selected hubs, enabling simplified routing and reduced operational duplication. Studies [e.g., Ceder, 2016; Klier and Haase, 2015] show that such designs can reduce total vehicle kilometers by up to 25% while maintaining acceptable levels of service and coverage. However, identifying optimal hub locations and balancing cost-efficiency with travel time performance remains a multi-objective and data-sensitive optimization problem, especially in medium to large urban networks.

Currently, several bus organizations are operating in the form of traditional destination-oriented systems. However, such systems are not in a well-defined framework, and these organizations continue to operate without regard to the concept of efficiency and system optimization, resulting in suboptimal service quality and increased operational expenditures. The existing traditional destination-oriented systems directly connect different locations point by point, where there are a large number of links that increase traffic congestion. Designing an efficient urban transportation system in the passenger transport sector is, therefore, a basic need for cities.

The hub location problem in public transport (PT) is therefore a key research area, focusing on strategically placing transport hubs to optimize service delivery and operational efficiency. Establishing centralized hubs where multiple routes converge can greatly enhance connectivity, reduce travel times, and improve overall service quality for passengers.

A hub-and-spoke network aggregates multiple flows at a single hub node where the high-volume of aggregated flows travel from one hub to another via a hub-hub link. With the use of the hub system, the direct link is divided into separate connections to increase access to public transportation vehicles such as buses. Using the hub-and-spoke network and dividing a city into different areas and multiple allocations to meet regional demands, the public transportation system with hub location can be useful in reducing costs while helping to minimize traffic congestion and pollution. Public transportation systems require a large investment; therefore, it is very important to minimize the costs of constructing facilities and transportation. In addition, the transfer time is also very important to keep passengers satisfied with the convenience of the system after the transfer. Taking these two goals together into account in a public transportation system can have beneficial results. Furthermore, pre-selecting potential hub locations based on priority criteria, rather than considering all

[4] www.itf-oecd.org/itf-transport-outlook-2021

[5] https://www.worldbank.org/en/publication

network nodes, can streamline the optimization process.

As mentioned above, there exist some challenges in this context such as identify optimal locations for new hubs while balancing objectives such as minimizing operational costs and reducing maximum passenger travel time. This study aims to provide an efficient model for urban public transportation using hub-and-spoke networks to determine hub locations and manage passenger flow, thereby minimizing total costs. Additionally, a separate objective function focuses on minimizing maximum transfer time through a time utility approach. Innovative in its approach to designing an urban bus transportation network that enhances fleet productivity and facilitates efficient passenger movement. The study employs the second objective function to improve the minimizing maximum travel time. Another novel aspect is the application of a MCDM method to pinpoint areas with the highest potential for hub development. The problem is formulated under specific assumptions and solved using the LR algorithm.

The following contributions are noteworthy:

1. Clarification of the Hub Location Problem: The introduction clearly defines the hub location problem in the context of PT, highlighting its importance for enhancing transportation efficiency and service quality.

2. Structured Two-Stage Approach: This study introduces a structured methodology that first uses TOPSIS, a multi-criteria decision-making tool, to pre-select potential hub locations based on demand, access time, and cost. These candidate nodes are then used in a bi-objective mathematical optimization model to determine the final hub configuration. This two-stage process enhances computational efficiency and reflects a realistic, decision-support-oriented approach suitable for large urban transport networks. Indeed, it presents a bi-objective approach that balances minimizing operational costs with reducing maximum passenger travel time.

3. Enhanced LR Techniques: Enhance LR techniques by incorporating adaptive progression for λ (Lagrangian multipliers) to improve efficiency in solving bi-objective hub location problems. This may involve dynamically adjusting penalties based on the algorithm's convergence behavior. We apply and tune a subgradient-based LR algorithm to efficiently solve the proposed bi-objective hub location model. While the LR technique itself is well-established, its application here is customized for the structure of our model, including the use of pre-selected potential hubs, time-based objectives, and complex routing constraints, enabling effective performance on large-scale instances.

4. Impact on Urban Mobility: The introduction highlights the broader implications of strategically placed hubs on urban mobility, explaining how effective hub establishment can improve connectivity, decrease travel times, and increase PT usage, contributing to more sustainable urban environments.

2. Literature Review

To present a more coherent synthesis of the existing work, the literature related to the hub location problem in public transport is categorized into five key themes: (1) metaheuristic approaches such as genetic algorithms, (2) exact and MIP-based models, (3) bi- or multi-objective optimization models, (4) uncertainty-handling models including fuzzy logic, and (5) real-world case studies.

2.1. Genetic Algorithm and Metaheuristic Approaches

Several studies use metaheuristic methods like Genetic Algorithms (GA) and Particle Swarm Optimization (PSO) to solve large-scale hub location or transit design problems. Parti et al. presented a hub-and-spoke network for bus transit with a heuristic approach [Parti et al. 2005]. They introduced an optimization model for the bus transportation network, which used the hub-and-spoke network and the genetic algorithm to generate high-frequency sensible routes and branched them to solve problems. Similarly, Hosapujari et al. provided an optimization model for the development of the hub-and-spoke network for bus transportation operations, which used a genetic algorithm to solve the problem. They noted that due to the classification of routes and the problems that existed in the network of destination-oriented routes, the hub-and-spoke network could be used to route buses in large cities and gain the economic benefits of this network [Hosapujari and Verma 2013] Daneshvar et al. introduced an intelligent transportation system to enhance routing in the goods distribution network. This system integrates meta-heuristic algorithms, specifically a clustering algorithm, along with genetic and particle swarm optimization techniques. The proposed algorithm was validated using the Aarhus-Denmark dataset, which includes urban traffic, meteorological, and urban area data. Results indicated that the method significantly reduced transmission costs, including service delays and overall moving expenses, outperforming existing meta-heuristic algorithms in the literature [Daneshvar et al. 2023].

2.2. MIP and Exact Optimization Models

Mixed-integer programming (MIP) models have been used extensively for hub location problems.

Nickel et al. presented the practical models for hub location in public transportation. The first model, called the PT model, sought to find the optimal route by creating hub-and-spoke edges and passing these flows through the edges. Their second model, known as GPT, was an extension of the PT model. Because directional edges were permitted between non-hub nodes, paths had to include at least one hub node. Besides, this model took into account the costs incurred by the creation of spoke edges [S. Nickel et al. 2001].

Gelare and Nickel presented a new MIP mathematical model for PT, called the Hub Location Problem in Public Transport (HLPPT), which serves as the basic model for future research [Gelareh et al. 2011]. In this model, there was no limit to the number of links in each origin and destination path. Also, the cost structure neither necessarily satisfied the triangle inequality nor any other special structure. Fallah-Tafti et al. develop a mathematical model for rapid transit networks using a hub-and-spoke framework, which includes stopovers (stations) at the hub and non-hub (spoke) alignments. This model enables the transshipment of flows among spoke nodes. Additionally, the network infrastructure is shaped by decisions regarding the locations of hub and spoke nodes and edges, determining the rapid transit lines and the routes for demand. The model also incorporates criteria for profit and service time [Fallah-Tafti et al. 2022].

2.3. Bi-Objective and Multi-Objective Models

Multiple studies recognize the conflicting goals in transport planning, especially minimizing cost versus time or environmental impact. Hub location models used in transportation networks usually deal with goals such as cost and time optimization. In some cases, a combination of several goals is used in the model depending on the expectations, as mentioned in the following examples. Costa et al. presented a bi-objective model for hub locations to minimize the total transportation costs and the maximum service time of hubs [da Graça Costa et al. 2008]. In a similar vein, Yaman and Elloumi introduced a bi-objective model for the transportation network, in which the authors considered the size of the routes to be limited. Their study focused on two objectives of minimizing network transmission time and minimizing total transmission costs [Yaman and Elloumi 2012]. In a study, Kaveh et al. presented a new bi-objective model of the multi-allocation hub location problem for the urban transportation network. They considered the demand to be dependent on each hub center, and their model could calculate the number of each type of vehicle between the two hubs. The first objective function of this model was to maximize the profit from the establishment of hub centers, and their second objective function was to minimize the total transmission time in the network [Kaveh et al. 2019]. Rashidi Kahag et al. introduced a multi-objective intermodal hub-location-allocation problem, where both origin and destination hub facilities are modeled as an M/M/m queuing system. The primary goals are to minimize total costs and system time. To address large-scale issues, they proposed the MOIWO algorithm, featuring an efficient chromosome structure and a fuzzy dominance method. The entropy-TOPSIS method demonstrates that MOIWO excels in simultaneously optimizing all metrics [Rashidi Kahag et al. 2019].

2.4. Fuzzy and Uncertainty-Handling Models

Given the inherent uncertainty in public transport parameters, researchers have developed fuzzy and robust models. Niakan et al. provided a multi-objective model to optimize the design of the hub network in uncertainty. Their goals were to minimize the maximum travel time, minimize total transfer costs, and maximize the minimum service reliability [Niakan et al. 2015]. Khodashenas et al. presented an integrated multi-depot vehicle routing problem (MDVRP) with simultaneous pickup and delivery (SPD) while accounting for unpredictable costs associated with pickup, delivery, and transfer. They employed the SCA algorithm to optimize package dimensions and utilized NSGA II and MOALO algorithms to minimize three objective functions: 1) total costs, 2) CO_2 emissions, and 3) maximum working hours for drivers. To tackle the model's inherent uncertainty, they introduced a novel fuzzy-robust box optimization (FRBO) technique. The research findings indicated that as the uncertainty rate increased from 0.5 to 0.7, greenhouse gas emissions rose by 1.11%, overall expenditures increased by 1.72%, and drivers' working hours decreased by 11.98% [Khodashenas et al. 2023].

2.5. Real-World Case Studies and Applications

Practical implementation of hub models adds critical value. Research carried out by Verma et al. presented the hub-and-spoke model to improve the productivity of the bus routes. They conducted their study in Bangalore city and realized significant savings in operator costs and carbon dioxide emissions, as well as a 19.17% reduction in the fleet on the hub-and-spoke network, compared to the destination-oriented systems [Verma et al 2015]. Badi et al., identified the optimal hub airport location in five North African countries using a hybrid grey-CODAS approach based on five main criteria. These criteria included airport pricing, infrastructure (both hard and soft), catchment area and landside access, along with factors like market conditions and airline partnerships. The results indicated that among the five airports evaluated—Cairo, Tripoli, Tunisia-Carthage, Algeria-Houari Boumediene, and Morocco-Mohammed V International Morocco was selected as the best option for establishing a hub airport in North Africa [Badi et al. 2023]. Another study by Klier and Haase proposed a new optimization method to design a PT network that considered demand as an internal factor. They attributed the demand to the routes created and their frequency and tested this approach for the German city of Dresden [Klier and Haase 2015]. Their computational experiments showed that this approach was applicable in real-world conditions. They offered to consider the factor of fare and its effect on the model (in this case, it is possible to build a model to maximize revenue in the design of the PT network). Cancela et al. demonstrated that mathematical programming can optimize public transportation bus routes and schedules, considering user and service provider interests, consumer behavior, and transportation/infrastructure limitations. They believed that the optimal solution obtained (bus routes and frequencies) was appropriate for both users and operators. With mathematical formulation, they brought the model closer to a real system through more realistic modeling (which did not exist in the literature before them) [Cancela et al. 2015].

Concerning the concept of hub system, Ustadi and Shopi sought to test the efficiency of the hub system in PT and provide in-depth explanations of the concept of hubs in public transportation for the northern regions of the Malaysian Peninsular. From the data analysis, people's willingness to use the concept of the hub in public transportation [Ustadi and Shopi 2016].

Kayisoglu and Akgun introduced a multiple allocation model of the tree of hubs location problem, which required a tree topology among the hubs and minimized the transportation cost of sending flows between OD pairs. Unlike most studies in the literature that assume a complete network with costs satisfying the triangle inequality for problem formulation, they defined the problem on non-complete networks and developed a modeling approach that did not require any specific cost and network structure. The proposed approach could provide more flexibility in modeling several characteristics of real-life hub networks [Kayışoğlu and Akgün 2021]. Concerning the hub-and-spoke model, Yinying Tang et al. sought to construct a hub-and-spoke network and adopt the "collecting and transportation" organization mode, based on the single distribution p-hub median problem. Their model was constructed by considering the characteristics of CR Express in terms of cost and time, and a LR heuristic algorithm was designed to solve the model built for CR Express transportation network [Tang et al. 2021]. Similarly, Wang et al. used the hierarchical hub-and-spoke transport network to model the transportation of agricultural products. They utilized a genetic algorithm with two layers of chromosome coding to select the optimal hubs, which ensured that all agricultural products were delivered on time while minimizing the total network cost [Wang et al. 2021].

From this thematic review, it is evident that significant progress has been made across various dimensions of hub location modeling including metaheuristic optimization, exact MIP approaches, bi-objective formulations, and uncertainty-handling models. However, a clear research gap exists in the integration of these components into a cohesive and scalable framework. Specifically:

Most existing studies either use metaheuristics or exact methods independently, but few leverage hybrid approaches like Lagrangian Relaxation to combine scalability with solution quality.

While multi-objective models are common, many do not incorporate a structured, data-driven pre-selection phase (such as MCDM) to guide the optimization.

Real-world case study applications remain limited, particularly for models combining bi-objective optimization with MCDM.

This study addresses these gaps by proposing a two-stage framework that (1) uses TOPSIS for efficient pre-selection of hub candidates, and (2) solves a bi-objective hub location model using an adapted Lagrangian Relaxation method. This contributes a structured and computationally feasible approach suitable for medium to large-scale public transport networks.

3. Methodology

3.1. Model Assumptions

The assumptions of the problem are as follows:

1. The bus transportation system of a city is being examined.
2. Demand and travel information is deterministic, fixed, and available. A subset of network nodes is pre-selected as potential hub locations.
3. Allocation can be multiple.
4. A discount factor of α will be considered ($0 < \alpha < 1$).

5. Only a hub-type origin (destination) is allowed to select a hub edge to exit the origin (reach the destination).
6. The choice of edge types on the path between origin and destination i and j depends on the state of i and j; That depending on whether both, neither, or only one of them is a hub node.
7. The capacity of the routes is unlimited.
8. Both points of a hub edge are hub nodes.

Given the growing importance of public transportation, it seemed logical to consider a model for the bus network in the transportation of passengers. Assumptions 2 and 8 consider the amount of passenger demand and travel time to be definite and the capacity of routes to be unlimited for convenience. However, in future research, we will define travel demand in terms of uncertainty and the limited capacity of routes. According to assumption 6, hub edges are unique. Assumptions 7 and 9 represent the explicit features of a point and a hobby edge. To create a hub, it is better to use the points in the network. Finally, the alpha discount coefficient is used according to most hub location models.

3.2. Mathematical Symbols

Table 1 displays the mathematical symbols utilized in the LP models.

Table 1. The mathematical symbols

Model parameters	
Symbols	**Descriptions**
c_{ij}	Cost of transfer between two regions i and j
t_{ij}	Time of transfer between two regions i and j
w_{ij}	Demand between the two regions i and j
α	Discount factor
F_k	Cost of creating a hub at point k
I_{kl}	Cost of creating an edge between two hubs k and l
ww	Weight of the first objective function in the merged objective function
$H(k)$	Set of potential hubs
Decision variables	
Symbols	**Descriptions**
$x_{ijkl} = 1$	If the optimal path from i to j traverses the hub edge $k-l$ and, 0 otherwise
$a_{ijk} = 1$	If the optimal path from i to j traverses the spoke edge $i-k$ while i is not hub and, 0 otherwise
$b_{ijk} = 1$	If the optimal path from i to j traverses the spoke edge $k-j$ while j is not hub and, 0 otherwise
$e_{ij} = 1$	If the optimal path from i to j traverses $i-j$ and either i or j is a hub and, 0 otherwise
$y_{kl} = 1$	For the hub-level variables $k < l$, if the hub edge $k-l$ is established, 0 otherwise
$hk = 1$	If k is used as a hub node, 0 otherwise
β	The maximum transmission time through different routes

3.3. Mathematical Model

In this study, the proposed model is formulated as a bi-objective optimization problem, where the first objective aims to minimize the total cost (including transportation, hub establishment, and edge construction), and the second objective minimizes the maximum travel time across all passenger flows. To solve this bi-objective model, we apply the weighted sum method, which scalarizes the two objectives into a single aggregated objective function (Equation 20). This approach is chosen for its simplicity, interpretability, and computational tractability. The parameter *ww* allows decision-makers to assign relative importance to cost versus travel time. While alternative methods such as ε-constraint or Pareto front generation could be used, the weighted sum method is more practical for solving medium and large-scale networks within a reasonable computational time.

The proposed linear programing (LP) model is as follows:

$$\min \sum_{i}\sum_{j\neq i}\sum_{k\epsilon H(k)}\sum_{l\epsilon H(k),l\neq k} \alpha w_{ij}\, c_{kl} x_{ijkl} + \sum_{i}\sum_{j\neq i}\sum_{k\epsilon H(k),k\neq i,j} w_{ij}\, c_{ik} a_{ijk}$$
$$\sum_{i}\sum_{j\neq i}\sum_{k\epsilon H(k),k\neq i,j} w_{ij}\, c_{kj} b_{ijk} + \sum_{i}\sum_{j\neq i} w_{ij}\, c_{ij} e_{ij} + \tag{1}$$
$$\sum_{k} F_k\, h_k + \sum_{k}\sum_{l>k} I_{kl}\, y_{kl}$$

$$\min \beta \tag{2}$$
$s.t.$

$$\sum_{l\neq i} x_{ijil} + \sum_{l\neq i,j} a_{ijl} + e_{ij} = 1 \qquad \forall\, i, j\neq i \;\&\; l\epsilon H(k) \tag{3}$$

$$\sum_{l\neq i} x_{ijlj} + \sum_{l\neq i,j} b_{ijl} + e_{ij} = 1 \qquad \forall\, i, j\neq i \;\&\; l\epsilon H(k) \tag{4}$$

$$\sum_{l\neq k,i} x_{ijkl} + b_{ijk} = \sum_{l\neq k,i} x_{ijlk} + a_{ijk} \qquad \forall\, i, j\neq i, k\neq i,j \;\&\; k,l\epsilon H(k) \tag{5}$$

$$y_{kl} \leq h_k\,, y_{kl} \leq h_l \qquad \forall\, k, l>k \;\&\; k,l\epsilon H(k) \tag{6}$$

$$x_{ijkl} + x_{ijlk} \leq y_{kl} \qquad \forall\, i, j\neq i, k, l>k \;\&\; k,l\epsilon H(k) \tag{7}$$

$$\sum_{l\neq k} x_{kjkl} \leq h_k \qquad \forall\, j, k\neq j \;\&\; k,l\epsilon H(k) \tag{8}$$

$$\sum_{k\neq l} x_{ilkl} \leq h_l \qquad \forall\, i, l\neq i \;\&\; k,l\epsilon H(k) \tag{9}$$

$$e_{ij} \leq 2 - \left(h_i + h_j\right) \qquad \forall\, i, j\neq i \tag{10}$$

$$a_{ijk} \leq 1 - h_i \qquad \forall\, i, j\neq i, k\neq i,j \;\&\; k\epsilon H(k) \tag{11}$$

$$b_{ijk} \leq 1 - h_j \qquad \forall\, i, j\neq i, l\neq i,j \;\&\; k\epsilon H(k) \tag{12}$$

$$a_{ijk} + \sum_{l\neq k,j} x_{ijlk} \leq h_k \qquad \forall\, i, j\neq i, l\neq i,j \;\&\; k,l\epsilon H(k) \tag{13}$$

$$b_{ijk} + \sum_{l\neq k,i} x_{ijkl} \leq h_k \qquad \forall\, i, j\neq i, k\neq i,j \;\&\; k,l\epsilon H(k) \tag{14}$$

$$e_{ij} + 2x_{ijil} + \sum_{l\neq i,j} x_{ijil} + \sum_{l\neq i,j} x_{ijlj} \leq h_i + h_j \quad \forall i, j\neq i \;\&\; l\epsilon H(k) \tag{15}$$

$$x_{ijkl}\left(t_{ik} + t_{kl} + t_{lj}\right) \leq \beta \qquad \forall i, j\neq i, k\neq i,j \;\&\; k,l\epsilon H(k) \tag{16}$$

$$a_{ijk}\left(t_{ik} + t_{kj}\right) \leq \beta \qquad \forall i, j\neq i, k\neq i,j \;\&\; k\epsilon H(k) \tag{17}$$

$$b_{ijk}\left(t_{ik} + t_{kj}\right) \leq \beta \qquad \forall i, j\neq i, k\neq i,j \;\&\; k\epsilon H(k) \tag{18}$$

$$x_{ijkl}, a_{ijk}, b_{ijk}, e_{ij}, h_k\,, y_{kl} \in \{0,1\}\,, \qquad \beta > 0 \tag{19}$$

The objective (1) is the total cost of transportation plus hub nodes and edges setup costs. The Objective (2) indicates the time utility when trying to minimize the maximum travel time through different routes. The constraints (3)- (5) are the flow conservation constraints. In (6), it is ensured that both end-points of a hub edge are hub nodes. The constraints (7) ensured that if a flow passes through more than one hub in the hub-level network, it traverses hub edge(s) connecting these nodes. In (8)- (9) it is ensured that only a flow with origin (destination) of hub type is allowed to select a hub edge to depart from origin (arrive to the destination). Constraints (10)- (12) ensure the spoke-edge structure of the network, enforcing that any flow using a spoke edge must be connected to at least one hub node. Specifically:

Constraint (10) ensures that if a flow uses a spoke edge from node i, node j must be a hub. Constraint (11) ensures the opposite: if a flow uses a spoke edge from node j, node i must be a hub. Constraint (12) ensures that spoke edges can only connect to hub nodes, not to non-hub nodes.

This is ensured by (13) and (14). Selection of edges on the path between origin and destination i and j depends on the status of i and j whether both, none or just one of them is a node. Constraints (13) and (14) ensure that for any flow between nodes i and j, if the path includes an intermediate node (i.e., a node other than the origin and destination), that node must be a hub. This maintains the structural logic of the hub-and-spoke network, where only hub nodes serve as transfer or relay points between flows. Constraint (15) governs the overall edge selection between origin and destination nodes. If both nodes are non-hubs, only direct non-hub edges can be used. If either node is a hub, hub-to-hub edges or spoke-to-hub edges must be used. This constraint reinforces the hub-and-spoke structure by preventing non-hub nodes from using hub-to-hub edges and ensuring that hub connectivity is maintained in the network. Relationships (16)- (18) indicate that the value of β is the highest value of transfer time through different paths. (19) Describes the type of decision variables used in the model.

3.4. Selecting a Set of Potential Hubs

In the first part of the model, several points should be selected as hubs from all available points using analysis and decision-making methods. These points are the same potential hubs that the model selects the hubs only from these points. Identifying potential locations based on basic data is valuable for further focus and qualitative analysis. The TOPSIS method, whose preference technique is based on the similarity of the ideal solution, was first introduced by Huang and Yun in 1981 [Hwang and Yoon 1981], and is one of the most popular and well-known MCDM methods for ranking options. The primary reasons for using TOPSIS over other MCDM methods like MABAC, MARCOS, and VIKOR in selecting potential hubs are as follows:

- ✓ Simplicity and Intuition: TOPSIS is conceptually simple, making it easy for decision-makers to understand the selection of alternatives close to the ideal solution and far from the negative ideal, enhancing its user-friendliness over other methods.
- ✓ Clear Ranking: TOPSIS offers a straightforward ranking of alternatives based on their proximity to the ideal solution, making it more intuitive than MABAC or MARCOS, which involve more complex calculations.
- ✓ Flexibility with Criteria Types: TOPSIS effectively accommodates both qualitative and quantitative criteria, making it versatile for various decision contexts, unlike some methods that have limitations on the types of criteria they can handle.
- ✓ Robustness to Weight Changes: TOPSIS is less sensitive to weight variations than methods like VIKOR, resulting in more stable decision outcomes. This robustness is especially important when weight assignments are uncertain.

Each MCDM method has its strengths and weaknesses, but TOPSIS is favored in many decision-making scenarios due to its advantages. Ultimately, the chosen method should align with the specific context and requirements of the decision.

In the TOPSIS method, the positive ideal solution is the solution that maximizes the profit of the criteria and minimizes the cost of the criteria, while the solution of the negative ideal

(also called the non-ideal solution) is the solution that maximizes the cost of the criteria and minimizes the profit of the criteria. The best alternative is the one that is closest to the positive ideal solution and at the same time the farthest from the negative ideal solution.
The steps and decision-making process of the TOPSIS method are as follows.

Step 1. Define the Problem and Criteria:

- Identify the decision-making issue and outline the alternatives for evaluation.
- Establish evaluation criteria, categorizing them as benefit (prefer higher values) or cost (prefer lower values).

Step 2. Construct a Decision Matrix D, with rows for alternatives and columns for criteria. Each element r_{ij} indicates the performance of alternative i under criterion j.

Step 3. Normalize the decision matrix: Normalize the decision matrix to remove units of measurement, usually achieved with the following formula.

$$n_{ij} = \frac{r_{ij}}{\sqrt{\sum_{i=1}^{m} {r_{ij}}^2}}$$

Here, n_{ij} is the normalized value, r_{ij} is the original value, and m is the number of alternatives

Step 4. Weigh the normalized decision matrix: Multiply each normalized value by its corresponding weight, $W_{n\times n}$, to account for the importance of each criterion.

$$V = N_D \times W_{n\times n}$$

Step 5. Determine the positive ideal and the negative ideal: Determine the positive ideal solution (PIS, A^+) and negative ideal solution (NIS, A^-):

$$PIS = A^+ = v_1, v_2, \dots, v_{n^+}, where\ v_{j^+} = max(v_{ij}) \text{ for benefit criteria and } \min(v_{ij}) \text{ for cost criteria}$$

$$NIS = A^- = v_1, v_2, \dots, v_{n^-}, where\ v_{j^-} = max(v_{ij}) \text{ for benefit criteria and } \min(v_{ij}) \text{ for cost criteria}$$

Step 6. Calculate the distance from the ideals: Euclidean distance can be used to calculate the distance.

$$d_i^{\,+} = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^{\,+} \right)^2} \quad , i = 1,2, \dots, m$$

$$d_i^{\,-} = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^{\,-} \right)^2} \quad , i = 1,2, \dots, m$$

Step 7. Calculate the proximity coefficient

$$CL_i^{\,+} = \frac{d_i^{\,-}}{d_i^{\,-} + d_i^{\,+}}$$

Step 8. Rank the options: Any option with a larger CL is better.

4. Model Validation

The model is solved by coding in GAMS software using Mandl's network data with the CPLEX solver. The Mandl's and Sioux Falls networks are widely used benchmark networks in transportation and network optimization. Mandl's network is a small, synthetic framework for testing algorithms related to traffic assignment and network flow, comprising nodes and directed edges that represent intersections and roads. In contrast, the Sioux Falls network is a larger, more complex benchmark based on the real city of Sioux Falls, South Dakota, used for studying traffic assignment models and evaluating transportation planning algorithms. Both networks are essential tools for researchers and practitioners to test and validate optimization algorithms, traffic simulation models, and various methodologies in transportation engineering and operations research [Mathew and Sharma 2009]. This network consists of 15 nodes and 21 links as shown in Figure 1 having total demand of 15570 person trips per hour.

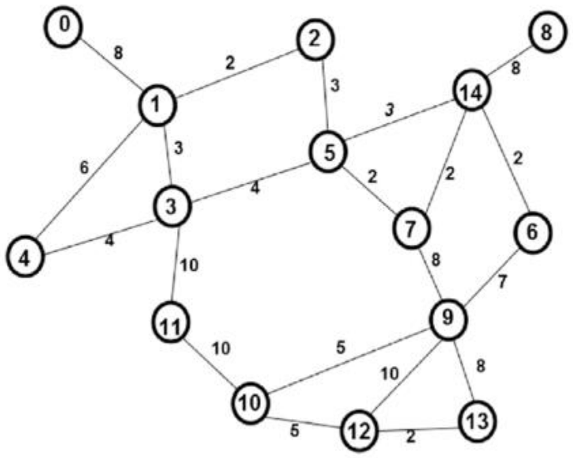

Figure 1. Mandl's network [Mathew and Sharma 2009]

In this study, it was assumed that creating a hub in each node would cost 10000 units. The demands between the points $w(i,j)$ in the network are derived from [9]. For the travel time parameter between nodes, a series of numbers are given on the network itself, which is the time of transfer between nodes that are connected with a direct link. For nodes that do not have a direct link, the transfer time is taken from the shortest path to these two nodes; that is, using the available links from the routes that connect the two nodes, the path time is calculated with the minimum displacement time [Buba and Lai 2018]. For the parameter of the transfer cost between different nodes, the following costs are considered in proportion to the time of transfer between nodes. Here, among nodes that do not have a direct link, the sum of the costs of the nearest possible path is calculated [Yadan et al. 2013].

The following parameter is defined as follows.

$I(k,l) = uniformint(500{,}1000)$

The LP (constraints (1) to (19)) method is used to solve the two-objective model. The merged target function is as follows. The weight of the first objective function or $\boldsymbol{ww}$ is 0.5.

$$z_f = ww\frac{z_1 - z_1^*}{z_1^*} + (1-ww)\,\frac{z_2 - z_2^*}{z_2^*} \qquad (20)$$

In the proposed model, the three parameters of demand, time and cost of transfer are the main data inform the model; Therefore, in order to select potential hubs according to the TOPSIS method, four criteria of demand, hub construction cost, minimum access cost and minimum access time have been considered, the first criterion is positive and the next criteria are negative. In this sense, the demand associated with a potential hub node refers to the sum of all origin and destination demands involving that node. Specifically, for each node i, we calculate:

$$Demand_i = \sum_j D_{ij} + \sum_j D_{ji}$$

Where D_{ij} is the travel demand from node i to node j. This captures the total passenger activity (both departing and arriving) at each node and serves as an indicator of the node's importance in the network. It does not include through-traffic (i.e., intermediate flows), as our aim at this stage is to assess hub candidacy based on local demand intensity.

The minimum access cost and minimum access time criteria represent the minimum cost/time for any other node in the network to reach the potential hub. For each node i, the access cost (or time) is calculated as:

$$\min_{j\neq i} C_{ji}, \quad \min_{j\neq i} T_{ji}$$

Where C_{ji} and T_{ji} are the cost and travel time from node j to node i.

In more detail, the “demand” criterion for each potential hub node is calculated as the total sum of originating and terminating passenger demand at that node (i.e., the sum of all demands from and to the node). This represents the node's overall passenger activity level. The “minimum access cost” and “minimum access time” are computed as the lowest transportation cost and travel time, respectively, from any other node in the network to the candidate node. These criteria aim to capture how easily other areas of the network can access a given potential hub, favoring more central and reachable locations.

The weights used in the TOPSIS method were determined based on expert input. Specifically, three academic experts in transportation systems were consulted to assess the relative importance of the four criteria (demand, hub construction cost, access cost, and access time). Their rankings were aggregated and normalized, resulting in a weight of 0.4 for demand and 0.2 for each of the remaining criteria. This reflects the central role of demand intensity in selecting hub locations, with the other criteria serving to refine the decision based on cost-efficiency and accessibility. The steps of the TOPSIS method have been implemented in Excel software. The final table for selecting potential hubs in Mandl's network will be as follows.

Table 2. Proximity coefficients of Mandl’s grid points for identifying potential hubs

Node	Final Score	Node	Final Score
0	0.0587	1	0.075
2	0.0669	3	0.073
4	0.0501	5	0.098
6	0.0728	7	0.0701
8	0.021	9	0.166
10	0.0599	11	0.022
12	0.0669	13	0.0568
14	0.0533		

The table clearly shows that points 1, 3, 5, and 9 have the highest scores and are proposed as potential hubs for the Mandle network. Considering the mentioned parameters, the model is solved for three different values of α-discount coefficient, and the results are shown in Table 3.

Table 3. The results of solving the model by Mandl's data

Number of nodes	α	Selected hubs	The optimal value of the first objective function	The optimal value of the second objective function
15	0.1	1,5,9	55220.2	33
15	0.5	1,5,9	56020.5	33
15	0.9	1,5,9	56647.45	33

The numerical example of the model presented with 8583 constraints and 5553 decision variables is solved in about 55 seconds in GAMS software. Given that the hubs must be selected from the potential points, the model selects the three points 1, 5, and 9 as hubs (which is created hub edge in pairs between these hubs) and assigns the other points to these three hubs to meet existing demands. As shown in the table, by changing the value of α, the set of selected hubs does not change. Also, considering that the second objective function is related to minimizing the maximum travel time; the optimal value of this objective function does not change with α change, but since the α-discount coefficient in the first objective function is available with a direct relationship, with increasing this parameter, the optimal value of the first objective function also increases.

An interesting result observed in Mandl's network is that the selected hubs (nodes 1, 5, and 9) and the second objective value (maximum travel time = 33) remained unchanged across all tested values of the α parameter. This stability suggests that hub selection in this case is robust to variations in cost weighting, and primarily driven by the network structure and demand distribution, rather than the cost scaling of inter-hub transfers. This behavior reinforces the strength of the model in identifying structurally optimal hub locations in smaller networks.

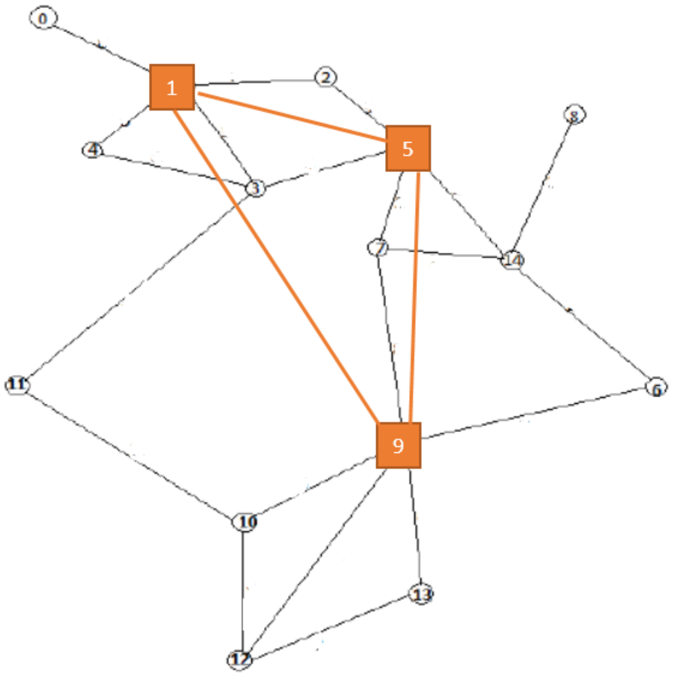

Figure 2. Mandl's network after solving the model

The basis of the first objective function is based on the diversity of flow transmission paths, for example, for the transfer of a passenger from node 10 to node 2, three types of routes are defined.

Table 4 illustrates examples of three possible routes between origin node 10 and destination node 2. The "Route" column lists the actual travel path as a sequence of nodes (e.g., 10–9–5–2). These paths reflect different ways of satisfying the hub-and-spoke structure: connecting a non-hub origin to a hub, transferring between hubs, and then reaching a non-hub destination. The labels x, a, and b refer to different edge types used in the route, where:

$x_{i,j,k,l}$: flow traverses a hub-to-hub segment from hub k to hub l,

$a_{i,j,k}$: flow begins at a non-hub origin i and enters hub k,

$b_{i,j,l}$: flow ends at a non-hub destination j after leaving hub l.

For instance, in Path 1, the route 10–9–5–2 implies that node 10 (non-hub) connects to hub 9, continues to hub 5, and finally reaches destination node 2 (also a non-hub). This route involves one spoke-to-hub, one hub-to-hub, and one hub-to-spoke link.

Table 4. Routes created to transfer from origin 10 to destination 2

Transfer Routes	Routes
Path 1 $\mathbf{x}_{(10-2-9-5)}$	10-9-5-2
Path 2 $\mathbf{a}_{(10-2-9)}$	10-9-7-5-2
Path 3 $\mathbf{b}_{(10-2-5)}$	10-9-7-5-2

In response to the numerical example, after assigning non-hub nodes to hubs, most routes are formed with direct links, and in routes where this is not possible, an attempt has been made to make the transfer with the least shift.

Figure 2 displays the network flow that indicates the hub location. Compared to other models, this one maintains stability despite changes in the α valuation. This issue was examined using a small sample from Mandl's network and dataset but can be effectively addressed with a larger dataset.

Figure 3 shows the schematic representation of the proposed hub allocation algorithm.

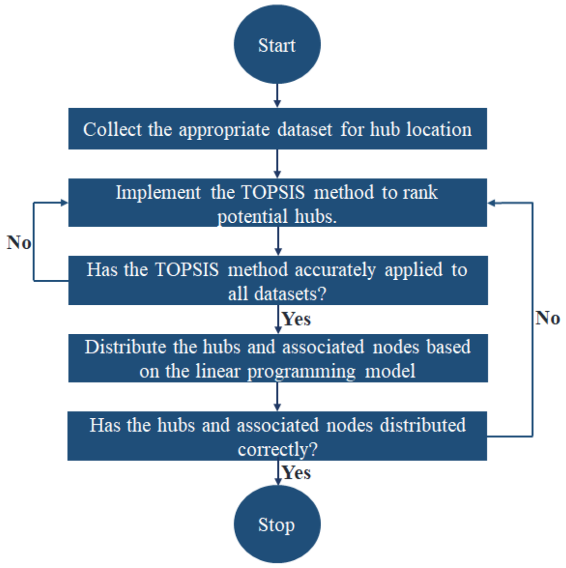

Figure 3. The schematic representation of the hub allocation algorithm

5. Lagrangian Relaxation Algorithm

Solving optimization problems in small and medium sizes is easily done by software such as GAMS, but by increasing the size of the problem, the solution speed is significantly reduced. Considering the structure of the proposed model and the time of solving numerical examples with different sizes, it was observed that with increasing the size of the model, the solution time in GAMS software increases exponentially, which shows the very complexity of the model, therefore, the LR algorithm is employed to solve the model for large-scale networks in a reasonable amount of time.

The LR method is one of the most widely used methods for solving constrained and difficult optimization problems, especially transportation planning problems. Although it has been suggested that the concept of LR was first used by Geoffrion in 1974 [Geoffrion 1974], several scientists before him used this algorithm. For example, in 1970 and 1971 Held and Karp [Held and Karp 1970□□Held and Karp 1971] used the concept for the traveling salesman problem. LR is commonly used in optimizing compounds to produce low bounds for minimization problems. In the field of location, Bilde and Krarup [Bilde and Krarup 1967] have used this algorithm to generate low bound in the problem of simple warehouse location. Diehr (1972) was the first researchers to use LR ideas to solve the p-median problem. Diehr proposed a heuristic algorithm that finds the ups and down bounds of the p- median problem [Diehr 1972]. To solve the inventory location problem, Chen et al. [Chen et al. 2011] used a LR algorithm to obtain a heuristic to find high bounds by fixing the output decision variables of the released problem in the main problem and the value of the resulting objective function was considered as the upper bound. The basis of the LR algorithm is to release the constraints that have the most complexity. These relaxed constraints are incorporated into the objective function with associated Lagrange multipliers, effectively penalizing violations. The purpose of using this algorithm is to reduce the solution time of the model and also to be able to get the suitable bounds for the problem [Fisher 2011].

If the general form of a problem is as follows.

$$\min f(x) \quad s.t. \tag{21}$$

$$Ax \geq b \tag{22}$$

$$Dx \geq d \tag{23}$$

$$x \in X \tag{24}$$

Now suppose that according to the model structure, the $Ax \geq b$ constraints have the most complexity and we want to release them, the problem form is as follows.

$$\begin{aligned} &\min f(x) + \lambda(b - Ax) \\ &\quad s.t. \\ &(23), (24) \\ &\quad \lambda \geq 0 \end{aligned} \tag{25}$$

When the $Ax \geq b$ constraint is violated, it means $b > Ax$, in which case the value of the expression $(b - Ax)$ is> 0, and multiplying this expression by the non-negative parameter λ increases the value of the objective function. This means that if a relaxed constraint is violated, the objective function value is penalized. The aim of objective function is minimizing the number of violated constraints.

Finding the constraints to be released is the first step. Usually, a constraint is selected for release, which firstly makes the model much more complex and secondly, the model is still able to produce a non-zero result by releasing this constraint. According to the structure of the proposed model and the literature review [Gelareh et al. 2011], constraints (8), (9), (13), (14) and (15) are selected to release the model. In fact, the constraints (8) and (9) enforce that only flows originating from or terminating at hub nodes are allowed to traverse hub-level edges. These constraints create a tight coupling between routing and hub assignment decisions. This cross-dependency significantly limits feasible routing combinations and increases the combinatorial complexity of the problem. Constraints (13) and (14) ensure that if a spoke edge connects to a non-hub node, the other endpoint must be a hub. These constraints govern the hierarchical nature of connectivity (hub-to-spoke), and again, they strongly interlink network topology with routing decisions. Constraint (15) specifies edge selection rules based on the hub status of the origin and destination. This is conditional logic embedded in constraint form, which is difficult for solvers to handle efficiently. It significantly increases branching in mixed-integer programming. By relaxing these constraints, we decouple hub assignment from routing logic, transforming the original tightly coupled mixed-integer program into a more decomposable structure. Constraints (3)- (5) (flow conservation) also contribute significantly to model complexity, but since these equations are the limitations of flow survival in the model and have a fundamental role in the output of the model, releasing them would render the model trivial (e.g., always yielding a zero solution), thus they cannot be relaxed. By releasing the specified restrictions, the model will look like this.

$$\begin{aligned} \min &\sum_{i}\sum_{j\neq i}\sum_{k\epsilon H(k)}\sum_{l\epsilon H(k),l\neq k} ww(\alpha w_{ij}\, c_{kl} x_{ijkl} + \sum_{i}\sum_{j\neq i}\sum_{k\epsilon H(k),k\neq i,j} w_{ij}\, c_{ik} a_{ijk} \\ &\sum_{i}\sum_{j\neq i}\sum_{k\epsilon H(k),k\neq i,j} w_{ij}\, c_{kj} b_{ijk} + \sum_{i}\sum_{j\neq i} w_{ij}\, c_{ij} e_{ij} + \\ &\sum_{k} F_k\, h_k + \sum_{k}\sum_{l>k} I_{kl}\, y_{kl} + \sum_{j,k\neq j\&l\in H(k)} \lambda_1 \left(\sum_{l\neq k} x_{kjkl} - h_k\right) + \\ &\sum_{i,l\neq i\&l\in H(k)} \lambda_2 \left(\sum_{k\neq l} x_{ilkl} - h_l\right) + \sum_{i,j\neq i,l\neq i,j\ \&\ k,l\epsilon H(k)} \lambda_3 \left(a_{ijk} + \sum_{l\neq k,j} x_{ijlk} - h_k\right) \\ &+ \sum_{i,j\neq i,k\neq i,j\&k,l\in H(k)} \lambda_4 \left(b_{ijk} + \sum_{l\neq k,i} x_{ijkl} - h_k\right) \\ &+ \sum_{i,j\neq i\&l\in H(k)} \lambda_5 \left(e_{ij} + x_{ijil} + \sum_{l\neq i,j} x_{ijil} + \sum_{l\neq i,j} x_{ijlj} - h_i - h_j\right) + (1 - ww)\beta \end{aligned} \tag{26}$$

s.t.

(3), (4), (5), (6), (7), (10), (11), (12), (16), (17), (18), (19)

In the next step, we need to assign a value to the Lagrangian coefficient parameter, λ. In this algorithm, treating λ as a decision variable would lead to an unbounded problem (infinite negative optimal value for a minimization problem).

Setting the primal value of λ to 0 allows us to solve the problem and obtain a lower bound. A straightforward approach to assign a value to this parameter is to create a loop that increases λ from 0 to 1 in specified increments (such as 0.1) and then select the value of λ that yields the best lower bound. However, it's important to note that the optimal λ value may not apply to all original problem constraints. A limitation of this method is that λ changes in fixed increments, implying that the rate at which the Lagrangian coefficient changes does not account for the degree of constraint violation. To improve this, we can link the Lagrangian coefficient to the violation level of the relaxed constraints, resulting in a more adaptive progression for λ. To implement this adaptive strategy, we can introduce a dynamic adjustment mechanism that modifies λ based on the magnitude of constraint violations observed during optimization. By evaluating the violation levels at each iteration, we can compute a related factor that indicates how much λ should be increased or decreased. For instance, if a certain constraint is significantly violated, we could increase λ more aggressively to penalize this violation, thereby encouraging the solution to shift towards feasibility. Conversely, if we observe minimal violations or even adherence to constraints, we could either slow down the increment of λ or potentially decrease it. This fosters a more responsive optimization process, as the adjustment of λ becomes contingent upon real-time feedback rather than a pre-defined set of increments.

Linking the Lagrangian coefficient to the extent of constraint violation, the increments of λ are as follows:

Step 1. Initialize the Lagrange multiplier vector (λ) and calculate an initial upper bound (UB) (e.g., from a feasible solution) and assume $LB^* = -\infty$.

Step 2. Solve the released problem and its optimal values, i.e. calculation x^* and LB.

Step 3. If LB $> LB^*$ then $LB^* = LB$.

Step 4. $\lambda^{(t)}$ the Lagrangian coefficient in t iterations and k is the rate of change of this parameter ($\theta = 0.5$).

$$\lambda^{(t)} = \lambda^{(t-1)} + k(b - Ax) \quad (27)$$

$$k = \theta \frac{UB - LB^*}{\sum_{i=1}^{n}(b_i - a_i x)^2} \quad (28)$$

$b - A$ = Violation of released restrictions,

$\sum_{i=1}^{n}(b_i - a_i x)^2 =$ Deviation rate of all released restrictions

Step 5. If there is no improvement in the lower bound after m iterations, we halve θ.

Step 6. Refer to Step 2.

In the subgradient optimization process, the initial step size parameter θ_0 is set as twice the initial duality gap, i.e., $\theta_0 = 2 \times (UB_0 - LB_0)$. At each iteration, the step size is reduced using $\theta_t = {\theta_0}/{t}$, and is halved if no improvement in the lower bound is observed over 10 consecutive iterations. The algorithm terminates when one of the following stopping condition is met: (i) a maximum of 300 iterations reached, (ii) the relative duality gap is less than 1%, or (iii) θ falls below 10^{-4}.

6. Computational Results

In this section, the results are compared using a mathematical model and the Lagrangian Relaxation algorithm in different dimensions. Finally, the two solution methods are compared based on solution time and performance.

Two small and medium-sized urban networks, Mandl and Sioux Falls, are considered to measure computational results. In Section 6.1, the results of the model solution were examined with Mandl's network data, and in this section, the model results for the Sioux Falls network are presented, and then the problem-solving algorithm will be discussed. Sioux Falls is the largest city in the state of South Dakota. The city's transportation network is also one of the most reputable networks where researchers review the results of their research. This network, as shown in Figure 4, has 24 stations and 76 communication links, which represent the arc numbers on the top of the network arcs [V. Mathew and Sharma 2009; Liu and Zhu □□□□□.To test the model on this network, demand, travel time, and transmission cost parameters specific to the Sioux Falls network are used (the data of this network is taken from the transportation networks section of GitHub[6] website). The amount of other parameters (fix cost of hub construction, discount coefficient and weight of objective functions) will be the same as those used for Mandl's network.

As usual before solving the model, it is necessary to identify the potential hubs of this network. Based on the TOPSIS decision-making method and the criteria mentioned in the previous network, nodes 3, 5, 9, 10, 14, 19, 22 and 24 are selected as potential nodes of this network after performing calculations in Excel. The results of solving the model with the Siux Falls network data and other parameters mentioned in GAMS software are given in Table 5.

Table 5. The results of solving the model by Sioux Falls data

Number of nodes	α	Selected hubs	The optimal value of the first objective function	The optimal value of the second objective function
24	0.1	3,5,9,10,14,19,22,24	5291981	16.4

In this network after solving, by constructing **5** hub centers, an attempt has been made to transfer the flow through the shortest routes. The model selected these **5** centers in such a way that the city's bus system has the lowest cost to meet the needs of the people. Compared to Mandl's network, because this network is more extensive (in addition to the number of points and more links, both the amount of demand between the points and the unit cost of the flow transfer are higher), the first objective function is much higher. Due to the schematic of the network and the location of the **5** selected hub centers, most transfers are done directly or with an intermediate hub, and in the case of routes that are not possible, the shortest routes have been used for these transfers. As mentioned, the time of solving model increases exponentially by increase the problem size. The Mandl's network with **15** nodes is solved in less than **1** minute, but the Sioux Falls network with 24 nodes takes about 35 minutes for the GAMS software to produce the output.

The discount coefficient α accounts for flow transfers through two intermediate hubs and is applied only to the costs in the first objective function, which is sensitive to α. The second objective function, related to time, is independent of α. Table 6 illustrates the effect of varying α from 0 to 1 in increments of 0.1.

Table 6. The varying α from 0 to 1 in increments of 0.1

1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0	**α**
56755	56647	56539	56431	56321	56194	56047	55901	55736	55455	55133	Z_1

6.1. Comparison of Results in Small and Medium Networks

To test the results of solving the mathematical model and algorithm, from the Mandle network three samples were produced and five samples were produced from the Sioux Falls network and solved by these methods. Also, the results of solving the mathematical model by CPLEX solver are recorded with the solving time and are compared with the results of solving the

1 https://github.com/bstabler/TransportationNetworks/tree/master/SiouxFalls

algorithms. In order to facilitate comparisons between these solutions, the percentage of relative deviation of each of the results is calculated according to the following equation:

$$GAP = \left(\frac{Sol - Sol_{Best}}{Sol_{Best}}\right) \times 100 \qquad (29)$$

In it, Sol_{Best} is the best lower bound from the LR algorithm, and Sol is the optimal solution value from GAMS (CPLEX). In the previous section, the concepts and how to use the LR algorithm on the model were explained. We considered the λ to be dependent on the violation amount of the relaxed constraints. According to the algorithm's definition, the initial upper bound can be a feasible solution, but due to the large volume of variables and equations in the model, this amount cannot be easily obtained by placement and manually. To do this, we have used a simple heuristic by solving the released problem and fixing the output decision variables in the main problem, and considering the value of the objective function as the initial UB. Indeed, obtaining a feasible UB from the solution of the relaxed problem is a standard heuristic step in Lagrangian relaxation frameworks. In our study, after solving the Lagrangian subproblem, we apply a repair-based heuristic, rather than directly fixing all decision variables from the relaxed solution x^*, due to the possibility of infeasibility when re-imposing the original constraints (particularly constraints (8), (9), (13), (14), and (15)).

Here's how the process works in more detail:

Initial solution from the relaxed problem:

We first solve the Lagrangian-relaxed model and obtain a solution x^*, which may violate some of the relaxed constraints.

- *Feasibility check:*

The relaxed solution is evaluated against the original model constraints. If it is already feasible, the objective function value is directly used as the UB.

- *Repair mechanism (in most cases):*

When infeasibilities are detected (e.g., flows using hub-edges without satisfying hub-node constraints), we apply a simple repair heuristic:

Edge-based repair: Invalid edge selections are removed or replaced with the closest feasible alternatives, i.e., rerouting through permitted hub/spoke paths.

- *Node reassignment:*

For spoke nodes wrongly using hub-level paths, we reassign their flows to valid hub paths based on proximity and demand.

Greedy adjustment: Infeasible routing patterns are iteratively adjusted using cost/time penalties as a guide until all original constraints are satisfied.

- *Objective reevaluation:*

After the repair phase, the revised solution is evaluated under the original objective function and constraints. This gives a valid feasible solution, which serves as the initial upper bound in the LR framework.

This repair heuristic ensures the practicality of the approach by bridging the gap between relaxed optimality and original feasibility, a strategy consistent with common practices in large-scale combinatorial optimization.

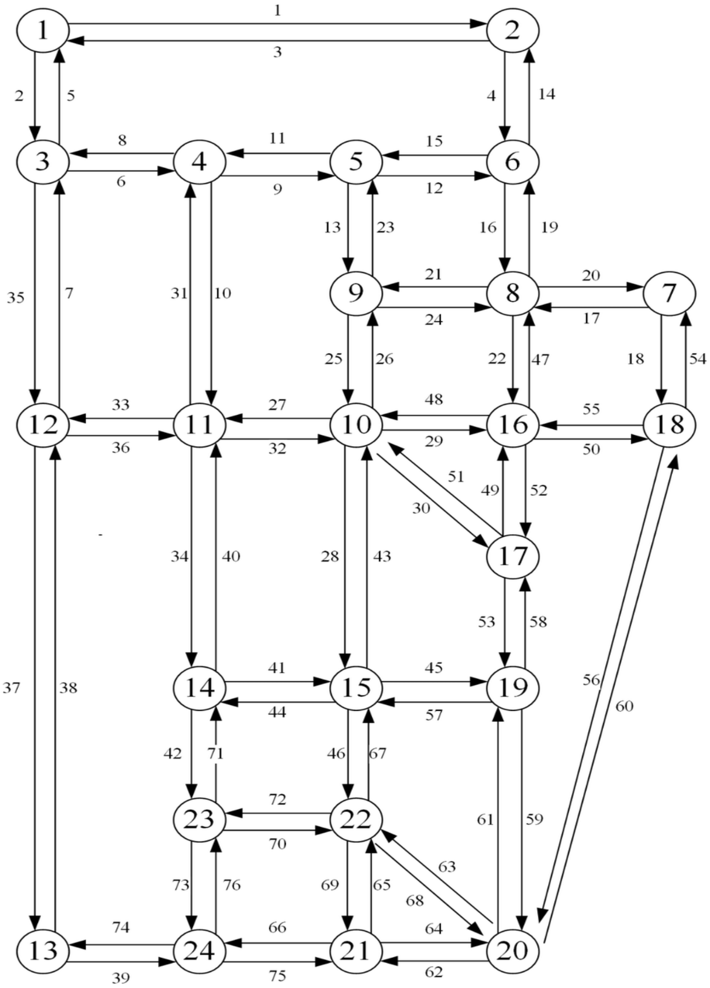

Figure 4. Sioux Falls network [V. Mathew and Sharma 2009; Liu and Zhu]

The results of comparing these two methods for the two networks are shown in Tables 7 and 8.

Table 7. The results of comparing the mathematical model and the LR algorithm (Mandl's Network)

Number of Nodes	Original Problem		Relaxed Problem		GAP(%)
	Time Solving(s)	Optimal Solution	Time Solving(s)	Best Lower Bound	
5	0.4	6487	42	6479.96	0.1
10	6	17720	245	17615	0.5
15	54	27743	672	25632	7.6
			Average		2.73

Table 8. The results of comparing the mathematical model and the LR algorithm (Sioux Falls Network)

Number of Nodes	Original Problem		Relaxed Problem		GAP(%)
	Time Solving(s)	Optimal Solution	Time Solving(s)	Best Lower Bound	
5	0.3	23814.5	53	23722	0.3
10	11	390911.5	238	385604	1.3
15	53	814214.7	422	812778.6	1.8
20	506	1595002	835	1483006	7.02
24	1988	2645999	2618	2462856	6.9
			Average		3.46

In Tables 7 and 8, the column labeled “Optimal Solution” represents the objective value obtained from the exact solution of the full model using GAMS/CPLEX. The column labeled “Best Lower Bound” refers to the best lower bound (LB^*) obtained from the Lagrangian Relaxation algorithm after all iterations. The relative performance of the LR algorithm is measured using the following GAP formula:

$$GAP\ (\%) = \frac{Optimal\ Solution - Best\ Lower\ Bound}{Best\ Lower\ Bound} \times 100$$

Since the problem is formulated as a minimization, this gap quantifies how close the LR-based solution is to the true optimum.

6.2. Comparison of Results in Large Networks

As mentioned, due to the high complexity of the model by increasing the size of the problem, solution speed is significantly reduced. The performance of the algorithm can be observed when GAMS software cannot solve the model in a reasonable time (for example, 2 hours). A grid network is used to solve large-scale models. The structure of these networks is such that they only have links with adjacent nodes, and the rest of the transfers are done step by step with the least distance. The structure of the PT network of many modern cities, like the city of Barcelona, is designed according to these networks.

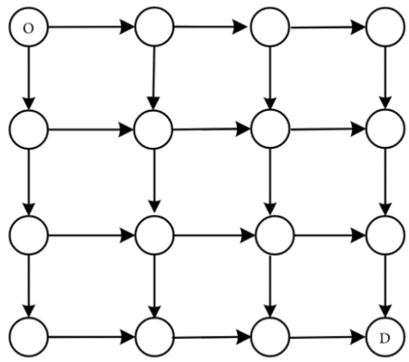

Figure 5. Schematic of Grid network

We have considered a 7*7 square network with 49 nodes and solved the released model with it. The problem parameters as follows.

$\alpha = 0.9,\ ww = 0.5,\ \ F_k = 10000$

$I(k,l) = uniformint(500{,}1000)$,

$w(i,j) = uniformint(100{,}300)$

The time parameters of time and the unit cost of flow transfer in the adjacent nodes are also considered in this way.

$t(i,j) = uniformint(5{,}10)$,

$c(i,j) = uniformint(2.5{,}5)$

The amount of time and cost of the other nodes that are not connected are also calculated in such a way that the minimum possible time and cost for the transfer is considered for them.

After 300 iterations, the algorithm reported the best lower bound for the problem as follows.

Table 9. The results of solving the model by grid networks data

Number of nodes	Number of potential hubs	Number of selected hubs	Time Solving(s)	The Best Lower Bound
49	16	9	3875	847489

The convergence process of the lower bound is also shown in the diagram below.

For the large-scale 49-node grid network, the LR algorithm successfully produced a valid lower bound in approximately 64.5 minutes (3875 seconds). This is a reasonable computational time given the model’s complexity and scale. As shown in Figure 6, the lower bound improves steadily across iterations, demonstrating the convergence behavior and stability of the subgradient-based multiplier update strategy. This result reinforces the suitability of the LR approach for solving large hub location problems where exact optimization is computationally infeasible.

6.3. Discussion of Computational Results

The proposed LR algorithm performs well in solving this problem, and with the release of several constraints, it is natural for the LR algorithm to provide a tight lower bound. In small and medium networks, according to the experience of using the LR algorithm on location problems and transportation networks, the average GAP rate obtained from both networks is acceptable. But in these examples, the solving time of the LR algorithm in them is much longer than the solving time of the model in GAMS.

In large networks where mathematical software is not able to solve them in a reasonable time, the proposed algorithm generates the appropriate bounds in a reasonable time for the problem. LR algorithm, on the small scale, provided the optimal solution with an acceptable GAP level, and on the large scale, it was able to obtain appropriate bounds, indicating the algorithm's suitable performance for these instances.

Figure 6. The lower bound obtained from the various iterations of the algorithm

Regarding solution time, we saw that for small examples the proposed algorithm has more solving time, but in large examples (which due to the complexity of the model software are not able to solve it in reasonable time), the algorithm was solved in a reasonable time, which shows the LR algorithm is effective for large-scale models.

An interesting and somewhat counter-intuitive observation is that for small instances (e.g., Mandl's network), the proposed LR algorithm required more computation time than solving the exact model using GAMS/CPLEX. This is primarily due to the iteration overhead of the subgradient method, which involves repeated solution of mixed-integer subproblems and adaptive Lagrange multiplier updates. Additionally, even though the problem size is small, the relaxed problem retains binary decision variables and thus does not become trivially solvable. In contrast, CPLEX efficiently exploits problem structure, cuts, and heuristics to solve small MILPs quickly. These findings reinforce the view that LR is most beneficial for large-scale problems, where exact methods become computationally infeasible, and where producing tight lower bounds is of greater value.

7. Conclusion

Public transportation is one of the most important aspects of society, and if the demand of travelers cannot be met within a certain period, negative societal impacts such as traffic congestion, air pollution, and increased transportation costs will be exacerbated. will be intensified. Therefore, in today's industrial world, meeting the needs of users, and meeting their demand is one of the most important and fundamental points to maintain urban stability.

In Hub and Spoke networks, fewer vehicles and less manpower are required for operations. These networks routinely coordinate and manage data related to schedules and routes, streamlining fleet management. Transfers occur through a central distribution center, with hubs serving as focal points. By implementing innovative techniques and leveraging the Hub and Spoke strategy, effective strategic management in public transportation can be achieved.

Various studies in hub location utilize methods such as multi-objective optimization, hybrid grey-CODAS, bi-objective optimization, and the MOIWO algorithm [Badi et al. 2023; Kaveh et al. 2019; Rashidi Kahag et al. 2019; Daneshvar et al. 2023]. These studies primarily focus on selecting optimal locations, with less emphasis on minimizing transfer times or addressing large hub location datasets. This study introduces a new bi-objective model that minimizes both costs and maximum transfer time. Additionally, the LR algorithm enables the proposed model to achieve a suitable lower bound solution efficiently. In fact, the study examined the location of hubs in the city bus transportation network, considering existing assumptions and variables to minimize transportation costs. In addition to cost reduction, we aimed to minimize the maximum transfer time to enhance passenger transfer efficiency. The LR algorithm was employed to solve the model, providing a good lower bound when the software could not solve it efficiently. To evaluate the algorithm's effectiveness, we solved the problem across various dimensions and compared the results of the algorithm with those of the mathematical model in small and medium dimensions. For larger cases where the mathematical model was impractical to solve, the algorithms delivered suitable solutions.

This study addressed the hub location problem using a bi-objective framework and enhanced Lagrangian Relaxation techniques to analyze impacts on urban mobility. This research introduced a novel bi-objective model that minimizes costs and transfer times in the context of public transport. Additionally, LR technique is used to solve the hub selection problem. Results indicate that the proposed method provides an acceptable and reliable solution. From a management perspective, the work highlights the importance of sustainability in public transport planning. By optimizing hub locations, cities can promote eco-friendly transport options, reduce carbon footprints, and align with broader sustainability goals, advocating for integrated transport solutions. These contributions significantly enhance the understanding and practical application of the hub location problem in public transport systems. By leveraging LR and focusing on bi-objective optimization, researchers can develop robust solutions that effectively address both operational efficiency and user needs. Each contribution can be tailored to specific contexts, increasing the relevance and applicability of the research findings.

One important direction for future research is the incorporation of capacity constraints. While this study assumed unlimited capacity for both hubs and routes to focus on structural optimization, real-world systems are almost always constrained by vehicle frequency, infrastructure throughput, and hub facility limits. Extending the model to account for these practical limitations will improve its applicability and realism, and may also affect optimal hub configurations and network structure significantly.

Future research opportunities should consider that this study focused on deterministic network data, which can vary over time. Subsequent studies could explore potential distributions for specific parameters to better reflect real-world uncertainties. The model did not account for hub or link capacity limitations, which could be addressed by incorporating specific capacity constraints. Future work could also develop efficient algorithms for large-scale problems. While this study selected potential hubs based on demand, cost, and time, future research should incorporate geographical conditions and urban factors, such as highways and physical constraints. Other decision-making approaches for hub selection could also be explored. Lastly, this model could be applied to urban transportation networks to compare its results with existing systems.

Another important avenue for future work is to explore the use of alternative MCDM methods for the hub pre-selection phase. While TOPSIS was selected in this study for its computational efficiency and intuitive output, comparing its results with methods such as VIKOR, PROMETHEE, or MARCOS could reveal variations in hub prioritization and their downstream impact on network optimization. Such comparative studies would contribute to a more comprehensive understanding of how decision-making techniques influence the quality and robustness of hub location planning.

The research faced limitations, including an inability to obtain real-world datasets. Due to the absence of a comprehensive centralized system for intra-urban hub location data, the authors were compelled to use Mandl's and Sioux Falls Network datasets.

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