Dynamic Analysis of Axially Beam on Visco - Elastic Foundation with Elastic Supports under Moving Load

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Abstract:

For dynamic analyses of railway track structures, the algorithm of solution is very important. For estimating the important problems in the railway tracks such as the effects of rail joints, rail supports, rail modeling in the nearness of bridge and other problems, the models of the axially beam model on the elastic foundation can be utilized. For studying the effects of axially beam on the elastic foundation, partial differential equations which represent the independent variables should be utilized because of the beams have infinite degrees of freedom. In this paper, solution algorithm and process of the axially beam on the elastic foundation by considering the elastic supports under moving load have been studied and equations have been analyzed as closed form. The beam model includes visco – elastic foundation and elastic supports conditions. For considering the beam element, axial force has been considered beside of shear and moment forces. The solution algorithm is that firstly the differential equations of beam on the elastic foundation with elastic supports are derived and then these equations are solved parametrically by using separation of variables and orthogonality properties of modes. This process and solution have been presented as closed form in this paper. This problem wasn't investigated in the technical literature. This model can be utilized for the most problems in the railway tracks. The advantage of this paper is presentation of algorithm and process of parametric solution for an axially beam on the visco - elastic foundation with elastic supports.

Keywords: Railway track, dynamic analysis, axially beam on elastic foundation, elastic supports

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1. Introduction

Dynamic analysis of structure has an important role in design of structures. The dynamic effects related to internal forces in structures while kinematic effects related to displacements and deformations caused by vibrations. Dynamic loads created by different sources. Dynamic loads can be caused by unbalanced masses in vehicles, wind or earthquake, waves due to explosions, move of rail vehicles and sea waves. In general, there are two types of dynamic loads that are cyclic and non-cyclic loads.

The simplest type of periodic loads is harmonic loads. Non-cyclic loads created by different sources such as blast, wind or earthquake. Many parameters can affect on the structures analysis. These parameters include the mass of the structure, degrees of freedom, stiffness and damping of structure.

The first step in analysis is calculation of motion equations for estimating the vibrations of a structure. These equations can be solved by numerical or analytical methods. Parameters such as displacement and stresses can be calculated by solving these equations.

There are three common methods in the formation of the motion equations which are: direct approach equilibrium, Hamilton and Lagrange methods.

Generally, structures in terms of degrees of freedom are classified as three cases that include one degree of freedom, several degrees of freedom and continuous systems.

Because of the beams have infinite degrees of freedom, the partial differential equations are used. This study is in field of continuous systems. One of the models for analyzing the structures is the beam on the elastic foundation. [Morfidis and Avramidis, 2002] investigated the beam element on an elastic foundation by taking two parameters. [Bogacz and Czyczuła, 2008] studied response of beam on viscoelastic foundation under a series of moving load.

[Akour, 2010] studied the nonlinear effect of beam on the elastic foundation. [Abu-Hilal, 2006] investigated the dynamic response of Euler Bernoulli beam due to a constant moving load. [Yang and Chang, 2009] obtained frequencies due to the moving load on the bridge [Clough and Penzien, 2003]. Also, Chopra [1995] and Paz and Leigh [2004] studied the dynamics of structures and earthquake engineering problems. Yang [1986] investigated the random vibration problems. [Fryba, 1999] studied the problems subjected to moving loads. Lalanne [2002] studied the problem related to random and mechanical vibrations.

Also, [Solnes,1997] investigated the process of stochastic and random vibrations. In the most research works, the support conditions have been considered as ideal. Therefore in the present study, the beam on the elastic foundation with elastic supports has been studied by utilizing the dynamics analysis and the application of the orthogonality properties of modes.

For considering the beam element, axial force has been considered beside of shear and moment forces. In this paper, firstly the differential equations of beam on the elastic foundation with elastic supports are obtained and then these equations are solved by using separation of variables and orthogonality properties of modes.

2. Algorithm and Problem-Solving Process

In order to study and analyze the problems, the algorithm must be presented. This algorithm is presented in Figure 1. This algorithm includes the problem modeling, free diagram of beam element, the differential equation of beam element, solving the differential equation by using separation of variables and solving the differential equation by using orthogonality properties of modes.

Also, the solution algorithm of differential equation by the method of separation of variables includes vertical displacement of beam in term of two functions with variables of “x” and “t”, the equation of free vibrations without effect of load, the equations in term of functions with two independent variables, solving two independent equations, applying boundary conditions at beginning and end of the beam, the determinant of a square matrix, frequency equation and coefficients of matrix.

Finally, the solution steps of the differential equation by using the orthogonality properties of modes include calculation of beam displacement as two functions with variables of “x” and “t”, use of the orthogonality properties of modes, calculation of damping as a combination of mass and stiffness and access to the equation with one degree of freedom.

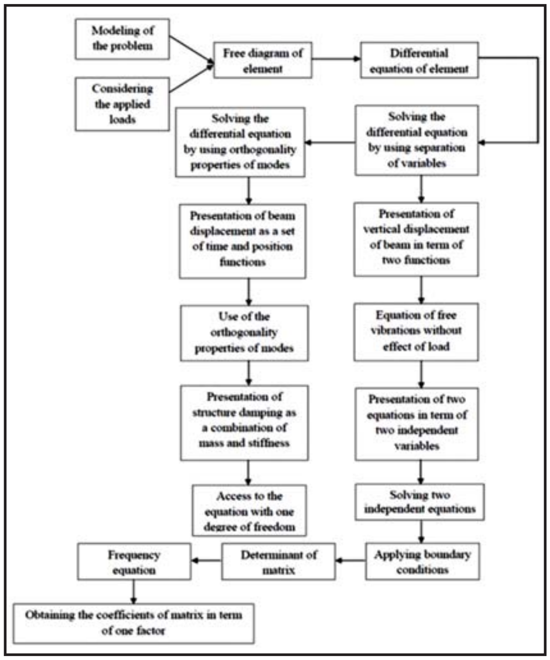

Figure 1. Algorithm and process of solution

Thus, the problem can be studied and investigated by using the above algorithm. In continue, the model of beam on the elastic foundation with elastic supports is presented.

3. Beam on the Elastic Foundation with Elastic Supports

For investigating the model of beam on the elastic foundation with elastic supports, a model is considered as a Figure 2. This model includes stiffness, damping, mass per unit length, elastic supports and so on.

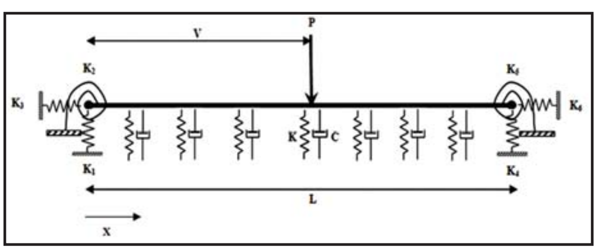

Figure 2. Beam on the elastic foundation with elastic supports

The considered beam is subjected to moving load P (x) as a following equation:

$$P(x) = P\delta(x - v) \tag{1}$$

In this equation, "v" and "P" are distance of load from the beginning of beam and amplitude of applied load respectively. Also, "δ" is the Dirac delta function. The differential equation of beam element is obtained with considering the beam element and free diagram of forces acting on the element (Figure 3).

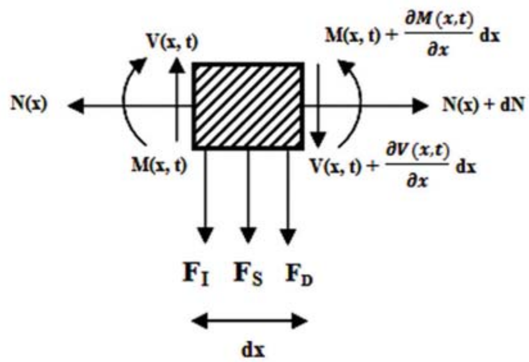

Figure 3. Free diagram of beam element

In this Figure, V (x, t), M (x, t) and N (x) are shear, moments and axial forces of element respectively. Also, F_I, F_S and F_D represent the inertia force, the stiffness force and damping force respectively.

By considering the free diagram of beam element, the equation of beam element resting on elastic foundation is presented as follows:

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 v(x,t)}{\partial x^2}\right]-\frac{\partial}{\partial x}\left[N(x)\frac{\partial v(x,t)}{\partial x}\right]+m(x)\frac{\partial^2 v(x,t)}{\partial t^2}$$
$$+C(x)\frac{\partial v(x,t)}{\partial x}+Kv(x,t)=-P\delta(x-v) \tag{2}$$

After achieving the partial differential equation of beam element, this equation is solved based on the following procedures.

4. Solving the Problem by using Separation of Variables

By using the separation of variables, the vertical displacement of beam, V (X, T), in terms of the variables "x" and "t" are considered as follows:

$$V(x,t)=\phi(x).Y(t) \tag{3}$$

Free vibration equation of the beam without the load effect is obtained as follows:

$$EI\phi^{IV}(x)Y(t)-N\phi^{II}(x)Y(t)+m(x)\phi(x)\ddot{Y}(t)$$
$$+C(x)\phi(x)\dot{Y}(t)+K\phi(x)Y(t)=0$$

By dividing the free equation by $\phi(x).Y(t)$, the beam equation is obtained as follows:

$$EI\frac{\phi^{IV}(x)}{\phi(x)}-N\frac{\phi^{//}(x)}{\phi(x)}+m(x)\frac{\ddot{Y}(t)}{Y(t)}+C(x)\frac{\dot{Y}(t)}{Y(t)}\phi(x)+K=0 \quad (5)$$

By dividing the above equation by EI, the following equation is obtained.

$$\frac{\phi^{IV}(x)}{\phi(x)}-\frac{N}{EI}\frac{\phi^{//}(x)}{\phi(x)}+\frac{K}{EI}=-\frac{m}{EI}\frac{\ddot{Y}(t)}{Y(t)}-\frac{C}{EI}\frac{\dot{Y}(t)}{Y(t)}\phi(x) \quad (6)$$

The above equations are separated as functions of two independent variables.

$$\frac{\phi^{IV}(x)}{\phi(x)}-\frac{N}{EI}\frac{\phi^{//}(x)}{\phi(x)}+\frac{K}{EI}=-\frac{m}{EI}\frac{\ddot{Y}(t)}{Y(t)}-\frac{C}{EI}\frac{\dot{Y}(t)}{Y(t)}\phi(x)=a^4 \quad (7)$$

Thus, two equations in terms of two variables are obtained. If the above equation is a constant, the above equation will have a solution.

Parameters of “a”, “b” and “g” are defined as follows:

$$a^4-\frac{K}{EI}=b^4$$
$$\frac{N}{EI}=g^2 \quad (8)$$

Therefore, two independent equations are obtained in terms of “x” and “t” which are:

$$\phi^{IV}(x)-g^2\phi^{//}(x)-b^4\phi(x)=0 \quad (9)$$

$$m\ddot{Y}(t)+C\dot{Y}(t)+EI\,a^4Y(t)=0 \quad (10)$$

The solution of first equation is considered as follows:

$$\phi^{IV}(x)-g^2\phi^{//}(x)-b^4\phi(x)=0$$
$$\lambda^4-g^2\lambda^2-b^4=0$$
$$\lambda^2=A$$
$$A^2-g^2A-b^4=0 \quad (11)$$
$$A_{1,2}=\frac{g^2\pm\sqrt{g^4+4b^4}}{2}$$
$$\lambda_{1,2,3,4}=\pm\sqrt{\frac{g^2}{2}\pm\left(\frac{g^4}{4}+b^4\right)^{1/2}}$$

If the parameters of “e” and “δ” are defined as follows:

$$\varepsilon=\sqrt{\frac{g^2}{2}+\left(\frac{g^4}{4}+b^4\right)^{1/2}}$$
$$\delta=\sqrt{\frac{g^2}{2}-\left(\frac{g^4}{4}+b^4\right)^{1/2}} \quad (12)$$

Therefore, the equation of $\phi(x)$ is obtained.

$$\phi(x)=A_1\cos(\delta x)+A_2\sin(\delta x)+A_3\sinh(\varepsilon x)+A_4\cosh(\varepsilon x) \quad (13)$$

Boundary conditions for shear and moment at the beginning and end of beam are defined as follows:

$$EI\,\phi^{///}(0)=K_1\phi(0)$$
$$EI\,\phi^{//}(0)=K_2\phi^{/}(0)$$
$$EI\,\phi^{///}(L)=K_4\phi(L) \quad (14)$$
$$EI\,\phi^{//}(L)=K_5\phi^{/}(L)$$

By applying the boundary conditions, matrix equation is obtained as follows:

In order to non-zero of coefficients, the determinant of a square matrix for this equation must be equal to zero that the frequency equation is reached. The determinant of the coefficient matrix with assuming “δ = a” and “ε = b” is presented in the appendix as parametrically (Equation 1A). Also, coefficients of matrix in terms of A_4 are presented in the appendix as parametrically (Table 1). The beam equation can be calculated by using the orthogonality properties of modes. The solution process is presented in continue.

5. Solving the Equations by using the Orthogonality Properties of Modes

For solving the equation, orthogonality properties of

$$\begin{bmatrix} K_1 & EI\,\delta^3 & -EI\,\varepsilon^3 & K_1 \\ EI\,\delta^2 & K_2\delta & K_2\varepsilon & -EI\,\varepsilon^2 \\ EI\,\delta^3\sin(\delta L)-K_4\cos(\delta L) & -EI\,\delta^3\cos(\delta L)-K_4\sin(\delta L) & EI\,\varepsilon^3\cosh(\varepsilon L)-K_4\sinh(\varepsilon L) & EI\,\varepsilon^3\sinh(\varepsilon L)-K_4\cosh(\varepsilon L) \\ -EI\,\delta^2\cos(\delta L)+K_5\delta\sin(\delta L) & -EI\,\delta^2\sin(\delta L)-K_5\delta\cos(\delta L) & EI\,\varepsilon^2\sinh(\varepsilon L)-K_5\varepsilon\cosh(\varepsilon L) & EI\,\varepsilon^2\cosh(\varepsilon L)-K_5\varepsilon\sinh(\varepsilon L) \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (15)$$

modes can be utilized. For this purpose, displacement of beam is defined based on the two variables of “x” and “t” as follows:

$$V(x,t)=\sum_{i=1}^{\infty}\phi_i(x).Y_i(t) \tag{16}$$

In result, the beam equation is obtained as follows:

$$\sum_{i=1}^{\infty}\frac{d^2}{dx^2}\left[EI\frac{d^2\phi_i(x)}{dx^2}\right]Y_i(t)-\sum_{i=1}^{\infty}\frac{d}{dx}\left[N(x)\frac{d\phi_i(x)}{dx}\right]Y_i(t)+\sum_{i=1}^{\infty}m(x)\phi_i(x)\ddot{Y}_i(t)$$
$$+\sum_{i=1}^{\infty}C(x)\phi_i(x)\dot{Y}_i(t)+\sum_{i=1}^{\infty}K\phi_i(x)Y_i(t)=-P(t)\delta(x-v) \tag{17}$$

Also, orthogonality properties of modes are considered as follows:

$$\int_0^L \phi_n(x)\phi_i(x)dx=0 \qquad \omega_i\neq\omega_n \tag{18}$$

$$\int_0^L \phi_n(x)m(x)\phi_i(x)dx=0 \qquad \omega_i\neq\omega_n \tag{19}$$

$$\int_0^L \phi_n(x)\frac{d^2}{dx^2}\left[EI\frac{d^2\phi_n(x)}{dx^2}\right]dx=\omega_n^2\int_0^L\phi_n^2(x)m(x)dx=\omega_n^2M_n \tag{20}$$

The structural damping is defined as follows:

$$C=a_0M+a_1K \tag{21}$$

Where “a_0” and “a_1” are coefficients of damping and consequently orthogonality properties are satisfied.

By multiplying both sides of equation (17) by $\int_0^L\phi_n(x)$ and by using the orthogonality properties of modes, the below equation is obtained.

$$\sum_{i=1}^{\infty}Y_i(t)\int_0^L\phi_n(x)\frac{d^2}{dx^2}\left[EI\frac{d^2\phi_i(x)}{dx^2}\right]-\sum_{i=1}^{\infty}Y_i(t)\int_0^L\phi_n(x)\frac{d}{dx}\left[N(x)\frac{d\phi_i(x)}{dx}\right]$$
$$+\sum_{i=1}^{\infty}\ddot{Y}_i(t)\int_0^L\phi_n(x)m(x)\phi_i(x)+\sum_{i=1}^{\infty}\dot{Y}_i(t)\int_0^L\left(a_0M_n+a_1\omega_n^2M_n\right)\phi_n(x)\phi_i(x)+\sum_{i=1}^{\infty}Y_i(t)\int_0^L K\phi_n(x)\phi_i(x)=-\int_0^L\phi_n(x)P(t)\delta(x-v) \tag{22}$$

By using the orthogonality property of modes, the below equation is obtained as follows:

$$Y_n\omega_n^2M_n+\ddot{Y}_nM_n+\dot{Y}_n(a_0M_n+a_1\omega_n^2M_n)+Y_nK_n=P_n\phi_n(v) \tag{23}$$

Parameters of above equation are considered as follows:

$$M_n=\int_0^L m(x)\phi_n^2(v)dx$$
$$K_n=\int_0^L k(x)\phi_n^2(v)dx \tag{24}$$
$$P_n(t)=\int_0^L P\delta(x-v)\phi_n(v)dx=P_n\phi_n(v)$$

In these equations, displacement is considered as follows:

$$V(x,t)=\sum_{n=1}^{\infty}\phi_n(x)Y_n(t) \tag{25}$$

Also, shear and moment of beam are calculated as follows.

$$M(x,t)=EI\frac{\partial^2 v(x,t)}{\partial x^2}$$
$$V(x,t)=EI\frac{\partial^3 v(x,t)}{\partial x^3} \tag{26}$$

6. Numerical Solutions in the Special Case

The beam equation can be considered as a simplified equation as follows:

$$\frac{d^4}{dx^4}X(x)=a^4X(x) \tag{27}$$

If it is assumed that only “k_1” is effective for the boundary conditions and other parameters are considered as unit, then the boundary conditions are considered as follows:

$$(1)\,(1)D^{(3)}(X)(0)-(K_1)X(0)=0,\ (1)\,(1)D^{(2)}(X)(0)=0,$$
$$(1)\,(1)D^{(2)}(X)(1)=0 \tag{28}$$

By solving the equation (27) based on the boundary conditions (28), the response is obtained as follows:

$$-\frac{1}{2}\frac{\_C1\left(-2K_1\sin(a)+a^3\cos(a)-a^3\sin(a)-e^{-a}a^3\right)e^{ax}}{a^3\left(2\sin(a)-e^a+e^{-a}\right)}$$
$$+\frac{1}{2}\frac{1}{a^3\left(2\sin(a)-e^a+e^{-a}\right)}\Big(\_C1\left(-2K_1\sin(a)\right.$$
$$\left.-e^aa^3+a^3\cos(a)+a^3\sin(a)\right)e^{-ax}\Big)$$
$$-\frac{1}{2}\frac{1}{a^3\left(2\sin(a)-e^a+e^{-a}\right)}\Big(\_C1\left(-2e^aK_1+2e^{-a}K_1\right.$$
$$\left.+2a^3\cos(a)-e^aa^3-e^{-a}a^3\right)\sin(ax)\Big)+\frac{1}{2}\_C1\cos(ax) \tag{29}$$

For this, the following parameters are considered.

$$a = \frac{\pi}{2}$$
$$C1 = 1 \quad (30)$$
$$K_1 = 1$$

So if "x = -π ... π" is considered on the horizontal axis, the response of equation can be plotted as follows:

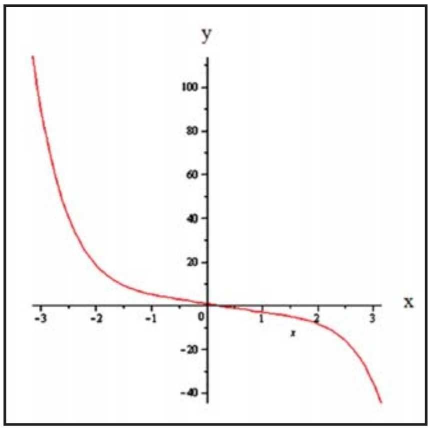

Figure 4. Response of equation based on "x" (the horizontal axis (x) is distance of load)

If "x =-π … π" and "k_1 = 0 … 10e20" are considered, the response of beam is presented as three-dimensional plot as follows:

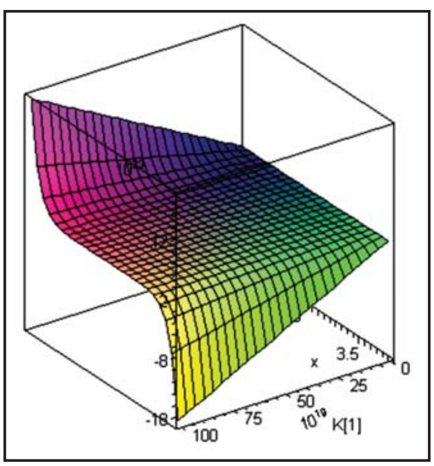

Figure 5. Response of equation based on "x" and "k_1"

As observed from the Figure, the response of equation approaches to negative values by increasing "k_1" that this matter intensifies by increasing "x". But in the range of negative values of "x", the response of equation approaches to positive values by increasing "k_1". But in the range of "x = 0", the effect of "k_1" is negligible. If it is assumed that "k_2" is effective in the boundary conditions beside of "k_1", the solution of the equation is obtained as follows:

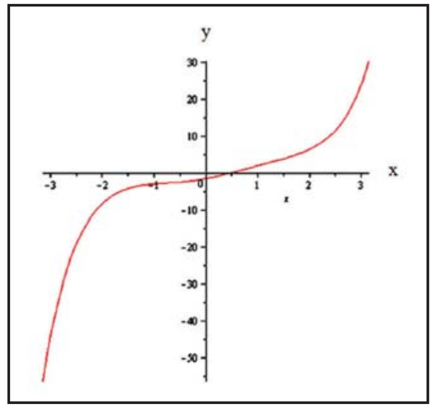

Figure6 . Response of equation based on «x» with considering "k_1" and "k_2"

As observed from the Figure, the response of equation approaches to positive values by increasing "x". Also, the response of equation approaches to negative values with decreasing "x".

7. Conclusion

For evaluating the important problems in the railway tracks such as the effects of rail joints, rail supports, rail modeling in the nearness of bridge and other problems, the models of the axially beam model on the elastic foundation can be utilized. The solution algorithm of the railway track model is important for analyzing these problems. For analyzing these problems, the partial differential equations that represent the independent variables are utilized. In the most research works, the beam with an ideal support conditions has been studied. Therefore in this paper, the effect of beam on the elastic foundation with elastic supports was studied by using the dynamics analysis of structures and orthogonality properties of modes. The solution algorithm and process of beam on the elastic foundation was presented as closed form. In this paper, firstly the differential equations of beam resting on the elastic foundation with elastic supports were obtained and then these equations were solved as closed form by using the separation of variables and the orthogonality properties of modes. The advantage of paper is that the algorithm and process of dynamic solution are presented for a beam on the elastic foundation with elastic supports. The main results of the paper are as follows:

* Obtaining the frequency equation and determinant of the coefficient matrix for the axially beam on the elastic foundation with elastic supports as parametrically.
* Obtaining the unknown coefficients of axially beam on the elastic foundation with elastic supports according to one factor as parametrically.
* For numerically analysis for special case i.e. if it is assumed only "k_1" is effective for boundary conditions, the response of equation approaches to negative values with increasing "k_1" that this matter intensifies with increasing "x". But in the range of negative values of "x", the response of equation approaches to positive values with increasing "k_1". But in the range of "x = 0", the effect of "k_1" is negligible.* Also if it is assumed that "k_2" is effective in the boundary conditions beside of "k_1", the response of equation approaches to positive values with increasing "x". Also, the response of equation approaches to negative values with decreasing "x".

8. References

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9. Appendix

D = -E^4*I^4*a^4*sin(a*L)*b^6*sinh(b*L)+k5*a^4*sin(a*L)*E^2*I^2*b^2*k4*sinh(b*L)-k5*a^2*sin(a*L)*k2*E^2*I^2*b^6*sinh(b*L)-E^2*I^2*a^2*k4*sin(a*L)*b^4*-k5*sinh(b*L)-2*k1*E^2*I^2*a^3*cos(a*L)*b^3*k5*cosh(b*L) - ....+ k5*a^2*sin(a*L)*k2*k1*E*I*b^3*cosh(b*L)+k5*a^2*sin(a*L)*k2*k1*k4*sinh(b*L)+k5*a*sin(a*L)^2*k4*E^2*I^2*b^5-k5*a*sin(a*L)^2*k4*k1*k2*b-k1*k5*a*cos(a*L)*E^2*I^2*b^5*cosh(b*L)-2*k4*cos(a*L)*E^2*I^2*a^3*k2*b^3*cosh(b*L)

Table 1. Matrix coefficients based on A_4

A_1	A_2	A_3
A4*(-k1*k4*k2*a-- ....-2*E^2*I^2*a^4*sin(a*L)*exp(b*L)*b^3*k2+2*k4*cos(a*L)*exp(b*L)*E*I*(a^3*k2*b-k1*E*I*b^3*k2*a	A4* (-k2*k1*E*I*b^4*exp(b*L)^2+ k2*k1*b*k4*exp(b*L)^2-.......... -2*E^2*I^2*a^4*sin(a*L)*exp(b*L)*b^3*k2+2*k4*cos(a*L)*exp(b*L)*E*I*a^3*k2*b (-k1*E*I*b^3*k2*a	A4*(-2*k1*k4*cos(a*L)* -exp(b*L)*k2*a+k1*k4*k2*a-.......... -2*E^2*I^2*a^4*sin(a*L)*exp(b*L)*b^3*k2+2*k4*cos(a*L)*exp(b*L)*E*I*a^3*(k2*b-k1*E*I*b^3*k2*a